ITERATIVE PROCEDURES FOR FINDING FIXED POINTS OF GIVEN MAPPINGS IN AN EFFECTIVE WAY

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คำสำคัญ: ปริภูมิไฮเพอร์โบลิกคอนเวกซ์เอกรูป, การส่งที่ไม่ขยายแบบเชิงเส้นกำกับชนิดผสม, การลู่เข้าแบบเข้ม, กระบวนการทำซ้ำนูร์แบบใหม่, จุดตรึง

บทคัดย่อ

กระบวนการทำซ้ำมีบทบาทที่สำคัญในการประมาณค่าจุดตรึงของการส่งไม่เชิงเส้น คุณสมบัติเชิงโครงสร้างของปริภูมิ เช่น ความนูนอย่างเข้ม และความนูนเอกรูป มีความจำเป็นอย่างมากสำหรับการพัฒนาทฤษฎีจุดตรึงแบบทำซ้ำในปริภูมิดังกล่าว ปริภูมิไฮเปอร์โบลิกมีลักษณะที่พบได้ในธรรมชาติและมีโครงสร้างทางเรขาคณิตที่หลากหลาย ซึ่งเหมาะสมในการหาผลลัพธ์ใหม่ๆ ในเชิงโทโพโลยี ทฤษฎีกราฟ การวิเคราะห์หลายค่า และทฤษฎีจุดตรึงเชิงเมตริก

วัตถุประสงค์แรกของวิทยานิพนธ์นี้คือเพื่อเสนอเทคนิควิธีการทำซ้ำนูร์แบบใหม่ สำหรับประมาณค่าจุดตรึงร่วมของสามการส่งในตัวแบบไม่ขยายเชิงเส้นกำกับ และสามการส่งนอกตัวแบบไม่ขยายเชิงเส้นกำกับในปริภูมิไฮเปอร์โบลิก อีกทั้งได้ให้ทฤษฎีบทการลู่เข้าแบบเข้มภายใต้เงื่อนไขที่เหมาะสมในปริภูมิไฮเปอร์โบลิกนูนเอกรูป

วัตถุประสงค์ที่สองคือการแนะนำและศึกษาบางทฤษฎีบทการลู่เข้าแบบเข้มสำหรับกระบวนการทำซ้ำ SP แบบผสมสำหรับสามการส่งในตัวแบบไม่ขยายเชิงเส้นกำกับ และสามการส่งนอกตัวแบบไม่ขยายเชิงเส้นกำกับในปริภูมิไฮเปอร์โบลิก นอกจากนี้ยังให้ตัวอย่างเชิงตัวเลขประกอบอีกด้วย

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ABSTRACT

Iterative schemes play a prominent role in approximating fixed points of nonlinear mappings. Structural properties of the underlying space, such as strict convexity and uniform convexity, are very much needed for the development of iterative fixed point theory in it. Hyperbolic spaces are general in nature and inherit rich geometrical structure suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory.

The first purpose of this dissertation is to propose a novel Noor iteration technique for approximating a common fixed point of three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings in hyperbolic spaces. Then, a strong convergence theorem under mild conditions in a uniformly convex hyperbolic space is established.

The second purpose is to introduce and study some strong convergence theorems for a mixed type SP-iteration for three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings in uniformly convex hyperbolic spaces. In addition to that, we provide an illustrative example.

The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

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CHAPTER I

INTRODUCTION

Iterative schemes play a prominent role in approximating fixed points of nonlinear mappings. Structural properties of the underlying space, such as strict convexity and uniform convexity, are very much needed for the development of iterative fixed point theory in it. Hyperbolic spaces are general in nature and inherit rich geometrical structure suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory.

In 1965, Browder [16], Göhde [33], and Kirk [50] started working on the approximation of fixed point for nonexpansive mappings. Firstly, Browder obtained fixed point theorem for nonexpansive mapping on a subset of a Hilbert space which is closed bounded and convex. Soon after, Browder [16] and Göhde [33] generalized the previous result from a Hilbert space to a uniformly convex Banach space. Kirk [50] utilized normal structure property in a reflexive Banach space to sum up the similar results. Recently, Dehici and Najeh [24] and Tan and Cho [91] approximated fixed point for nonexpansive mappings in Banach space and Hilbert space.

Hereafter, so many researchers came forward with different notions and enhanced this mapping with great improvement, there is a vast literature on the generalizations, extensions of the obtained results and several new concepts on nonexpansive mappings. In 1980, Gregus [34] generalized the work of Kannan [47] and joined the ideas of nonexpansive and Kannan mappings to obtain a unfamiliar class which is known as the Reich nonexpansive mappings. In 2008, Suzuki [82] proposed a different class of mappings which is known as Suzuki's generalized nonexpansive mapping. Recently, Ali et al. [10] proved some weak and strong convergence results using a three-step iterative scheme for Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces.

In 2011, Aoyama and Kohsaka [11] introduced a generalization of nonexpansive mappings known as α -nonexpansive mapping and obtained some results for this type of mappings.

Recently, Pandey et al. [67] proposed a different extension of nonexpansive mappings which contains α -nonexpansive and Suzuki generalized nonexpansive mappings named as generalized α -Reich-Suzuki nonexpansive mappings and obtained interesting results containing this kind of mappings.

Numerical reckoning of nonlinear operators is very fascinating research problem of nonlinear analysis. However, it is not an easy task to find the fixed points of some operators. To overcome this kind of problems so many iterative procedures have been evolved over the time. Mann [58], Ishikawa [42] and Halpern [39] are three basic iterative algorithms utilized to approximate the fixed points of nonexpansive mappings.

After getting motivation by the above iterative schemes, several researchers constructed many algorithms to approximate fixed points of numerous nonlinear mappings. A few of them are Noor iteration [62], Agarwal et al. [2], Abbas and Nazir iteration [1], Thakur New iteration [95], Picard-S iteration [36], normal-S iteration [37, 38], Ullah and Arshad (M) iteration [96], Garodia and Uddin [26] and many others.

Fixed point theory in partially ordered metric spaces has been initiated by Ran and Reurings [72] for finding application to matrix equation. Nieto and Lopez [61] extended their results for nondecreasing mapping and presented an application to differential equations. Recently, Song et al. [81] extended the notion of α -nonexpansive mapping to monotone α -nonexpansive mapping in order Banach spaces and obtained some existence and convergence theorem for the Mann iteration (see also [14] and the reference therein). Motivated by works of Suzuki [82], Aoyama and Kohsaka [11], Dehaish and Khamsi [14], and Song et al. [81], Pant and Shukla obtained existence results in ordered Banach space for a wider class of nonexpansive mappings [68, 69]. There are many mathematicians who worked on weak and strong convergence of nonexpansive mappings and its generalizations by using one step, two step, and multistep iteration process ([56, 81, 90]).

The class of asymptotic nonexpansive mappings has been extensively studied in fixed point theory since the publication of the fundamental papers [30]. Kirk and Xu [51] studied the asymptotic nonexpansive mapping in uniformly convex Banach spaces. Their result has been generalized by Hussain and Khamsi [41] to metric spaces. Khamsi and Kozlowski [49] extended their result to modular function spaces. In almost all papers, authors do not describe any algorithm for constructing fixed points for the asymptotic nonexpansive mapping. Ishikawa [42] and Mann [58] iterations are two of the most popular methods to check that these two iterations were originally developed to provide ways of computing fixed points for which repeated function iteration failed to converge. Espinola et.al [25] examined the convergence of iterates for asymptotic pointwise contractions in uniformly convex metric spaces. Kozlowski [53] proved convergence to a fixed point of some iterative algorithms applied to asymptotic pointwise mappings in Banach spaces. In [15], the authors discussed the convergence of these iterations in modular function spaces. In a recent paper [23], the authors investigate the existence of a fixed point of asymptotic pointwise nonexpansive mappings and study the convergence of the modified Mann iteration in hyperbolic metric spaces. It is well known that the iteration processes for generalized nonexpansive mappings have been successfully used to develop efficient and powerful numerical method for solving various nonlinear equations and variational problems.

Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces have been obtained in literature, for example, see [6, 7, 8, 9, 22, 27, 29, 43, 44, 65, 66, 83, 85, 84, 86, 87]. Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is easier working with Banach space due to its convex structures. However, metric space do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi [88] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [31, 52, 74]). Although the class of hyperbolic spaces defined by Kohlenbach [52] is slightly restrictive than the class of hyperbolic spaces introduced in [31], it is however, more general than the class of hyperbolic spaces introduced in [74]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [52]. Some other examples of this class of hyperbolic spaces includes Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and R-trees, see [12, 24, 31, 32, 52, 74].

The class of hyperbolic spaces, nonlinear in nature, is a general abstract theoretic setting with rich geometrical structure for metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory. Fixed point theory and hence approximation techniques have been extended to hyperbolic spaces (see [3, 4, 5, 13, 75, 76, 77] and references therein).

Recently, various fixed-point iteration processes for nonexpansive mappings have been studied extensively by many authors [46, 71, 59, 79, 89].

In 1972, Goebel and Kirk [30] introduced the class of asymptotically nonexpansive self-mappings. They proved that if $\mathcal{K}$ is nonempty closed convex subset of a real uniformly convex Banach space and $\mathcal{T}$ is an asymptotically nonexpansive self-mapping on $\mathcal{K}$, then $\mathcal{T}$ has a fixed point.

In 1991, Schu [78] introduced the following modified Mann iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}^n x_n, \quad n \geq 1, \tag{1.0.1}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schu's iteration process (1.0.1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces; see, e.g., [19, 21, 57] and the references therein.

Recall that a subset K of space X is said to be a retract if there exists a continuous mapping $\mathcal{P}: X \to \mathcal{K}$ such that $\mathcal{P}x = x$, $\forall x \in \mathcal{K}$. $\mathcal{P}: X \to \mathcal{K}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. If P is a retraction, then $x = \mathcal{P}x$ for all x in the range of $\mathcal{P}$. We refer to [17, 73, 32] for more details.

For any nonempty subset $\mathcal{K}$ of a real metric space (X, d), let $\mathcal{P}: X \to \mathcal{K}$ be a nonexpansive retraction of X onto $\mathcal{K}$. Then, $\mathcal{T}: \mathcal{K} \to X$ is said to be an asymptotically nonexpansive nonself-mapping (see [20]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}x, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}y) \leq k_n d(x, y) \tag{1.0.2}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$. We denote by $(\mathcal{P}\mathcal{T})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping.

For asymptotically nonexpansive nonself-mappings Chidume, Ofoedu, and Zegeye [20] studied the following iterative sequence

$$x_{n+1} = \mathcal{P}((1-\alpha_n)x_n + \alpha_n \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}x_n) \tag{1.0.3}$$

to approximate some fixed point of $\mathcal{T}$. They obtained a convergence theorem under suitable conditions in real uniformly convex Banach spaces. If $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping. Hence, (1.0.3) reduces to (1.0.1).

In 2006, Wang [97] considered the following iteration process which is a generalization of (1.0.3) (see also [93]),

$$\begin{aligned} y_n &= \mathcal{P}((1-\beta_n)x_n + \beta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n), \\ x_{n+1} &= \mathcal{P}((1-\alpha_n)x_n + \alpha_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}y_n), \ n \geq 1, \end{aligned} \tag{1.0.4}$$

where $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). They obtain a strong convergence theorem under weak restrictions imposed on the control parameters.

In 2012, Guo, Cho and Guo [35] further studied the following iteration scheme

$$\begin{aligned} x_n &= \mathcal{P}((1-\beta_n)\mathcal{S}_2^n x_n + \beta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n), \\ x_{n+1} &= \mathcal{P}((1-\alpha_n)\mathcal{S}_1^n x_n + \alpha_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n), \ n \geq 1, \end{aligned} \tag{1.0.5}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). Weak and strong convergence theorems of common fixed points of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and $\mathcal{T}_2$ were obtained.

Very recently, Jayashree and Eldred [45] introduced and studied the following mixed type iteration scheme in a uniformly convex hyperbolic space and prove some strong convergence theorems for mixed type asymptotically nonexpansive mappings:

$$\begin{aligned} v_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n, \alpha_n)), \\ u_{n+1} &= \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \beta_n)), \ n \geq 1, \end{aligned} \tag{1.0.6}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are two asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to X$ are two asymptotically nonexpansive nonself-mappings, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). Several papers have studied fixed points using two-step mixed type iterative schemes in a uniformly convex hyperbolic space (see [94]).

Another classical iteration precess was introduced by Noor [62] which is formulated as follows: $x_1 = x \in \mathcal{K}$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n \mathcal{S} x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n \mathcal{S} z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \mathcal{S} y_n, \ n \geq 1, \end{aligned} \tag{1.0.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1]. Such iterative method is called Noor iteration. Because of its simplicity, the method (1.0.7) has been widely utilized to solve the fixed point problem, and as a result, it has been enhanced by many works, as seen in [47, 63, 64, 72].

Glowinski and Le Tallec [28] employed three-step iterative approaches to find solutions for the problem of elastoviscoplasticity, eigenvalue computation and the theory of liquid crystals. In [28], it was shown that the three-step iterative process yields better numerical results than the estimated iterations in two and one steps. In 1998, Haubruge, Nguyen and Strodiot [40] studied the convergence analysis of three-step methods of Glowinski and Le Tallec [28] and applied these methods to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. As a result, we conclude that the three-step approach plays an important and substantial role in the solution of numerous problems in pure and applied sciences.

As reviewed, it is therefore the main objectives in this dissertation to introduce and study new type of iterative procedures for given mappings in an effective way. A sufficient conditions for convergence of such iterations to a common fixed point of mappings under our setting are also established. Furthermore, we then establish strong convergence theorems under some mild conditions in a uniformly convex hyperbolic space.

The results presented here extend and improve some related results in the literature.

CHAPTER II

PRELIMINARIES

2.1 Metric Spaces, Linear spaces, Normed spaces and Banach spaces

Now, we recall some well known concepts and results.

Definition 2.1.1. [54] A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$(triangle inequality).

Definition 2.1.2. [54] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or, simple} x_n \to x$$

we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 2.1.3. [54] A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if for every $\epsilon > 0$ there is an $N(\epsilon) \in N$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$.

Definition 2.1.4. [54] A metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.5. [54] Every convergent sequence in a matric space is a Cauchy sequence.

Theorem 2.1.6 *[60] Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.*

Definition 2.1.7. [60] Let X be a matric space and A be any nonempty subset of X. For each x in X, the distance $d(x, A)$ from x to A is $\inf\{d(x, y)|y \in A\}$.

Definition 2.1.8. [60] Let X be a linear space (or vector space). A norm on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:
(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\alpha x\| = |\alpha|\|x\|$,
(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).
The ordered pair $(X, \|\cdot\|)$ is called a normed space or normed vector space or normed linear space.

Definition 2.1.9. [60] Let X be normed space. The metric induced by the norm of X is the metric d on X defined by the formula $d(x, y) = \|x-y\|$ for all $x, y \in X$. The norm topology of X is the topology obtained from this metric.

Definition 2.1.10. [60] A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a **Banach space** or B-space or complete normed space if its norm is a Banach norm.

Definition 2.1.11. [52] A **hyperbolic space** $(X, d, \mathcal{H})$ is a metric space (X, d) together with a mapping $\mathcal{H} : X \times X \times [0, 1] \to X$ satisfying

$$(H1) : d(z, \mathcal{H}(x, y, \beta)) \leq (1 - \beta)\, d(z, x) + \beta d(z, y),$$

$$(H2) : d(\mathcal{H}(x, y, \beta)), \mathcal{H}(x, y, \gamma) = |\beta - \gamma|\, d(x, y),$$

$$(H3) : \mathcal{H}(x, y, \beta) = \mathcal{H}(y, x, (1 - \beta)),$$

$$(H4): d(\mathcal{H}(x,z,\beta),\mathcal{H}(y,w,\beta)) \leq (1-\beta)\, d(x,y) + \beta d(z,w)$$

for all $x, y, w, z \in X$ and $\beta, \gamma \in [0,1]$.

A subset $\mathcal{K}$ of a hyperbolic space X is convex if $\mathcal{H}(x,y,\beta) \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\beta \in [0,1]$.

Recall that a hyperbolic space $(X, d, \mathcal{H})$ is said to be

(i) strictly convex [88] if for any $u, v \in X$ and $\beta \in [0,1]$, there exists a unique element $z \in X$ such that $d(z,x) = \beta d(x,y)$ and $d(z,y) = (1-\beta)d(x,y)$;

(ii) uniformly convex [80] if for all $x, y, w \in X$, $r > 0$ and $\epsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d(\mathcal{H}(x,y,\frac{1}{2}),x) \leq (1-\delta)r$ whenever $d(x,w) \leq r, d(y,w) \leq r$ and $d(x,y) \geq \epsilon r$.

Recall that a mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$ is called modulus of uniform convexity. We call η-monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [55]).

Let (X, d) be a metric space, and let $\mathcal{K}$ be a nonempty subset of X. We denote the fixed point set of a mapping $\mathcal{T}$ by

$$F(\mathcal{T}) = \{x \in \mathcal{K} : \mathcal{T}x = x\}$$

and

$$d(x, F(\mathcal{T})) = \inf\{d(x,p) : p \in F(\mathcal{T})\}.$$

A self-mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be

(i) nonexpansive if $d(\mathcal{T}x, \mathcal{T}y) \leq d(x,y)$ for all $x, y \in \mathcal{K}$.

(ii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq k_n d(x, y)$ for all $x, y \in \mathcal{K}$ and $n \geq 1$.

(iii) uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$d(\mathcal{T}^n x, \mathcal{T}^n y) \leq \mathcal{L} d(x, y)$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$.

From the above definitions, one clearly sees that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1$, $\forall n \geq 1$. Both nonexpansive mappings and asymptotically nonexpansive mappings are Lipschitzian continuous. To be more precise, each nonexpansive mapping is $\mathcal{L}$-Lipschitzian and each asymptotically nonexpansive mapping is uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup\limits_{n \in \mathbb{N}} \{k_n\}$.

Recall that a subset $\mathcal{K}$ of space X is said to be a retract if there exists a continuous mapping $\mathcal{P} : X \to \mathcal{K}$ such that $\mathcal{P}x = x$, $\forall x \in \mathcal{K}$. $\mathcal{P} : X \to \mathcal{K}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. If $\mathcal{P}$ is a retraction, then $x = \mathcal{P}x$ for all x in the range of $\mathcal{P}$. We refer to [17, 73, 32] for more details.

For any nonempty subset $\mathcal{K}$ of a real metric space (X, d), let $\mathcal{P} : X \to \mathcal{K}$ be a nonexpansive retraction of X onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to X$ is said to be an asymptotically nonexpansive nonself-mapping (see [20]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} x, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1} y) \leq k_n d(x, y) \tag{2.1.1}$$

for all $x, y \in \mathcal{K}$ and $n \geq 1$. We denote by $(\mathcal{P}\mathcal{T})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping.

Lemma 2.1.12 *[92] Let* $\{a_n\}$,$\{b_n\}$ *and* $\{c_n\}$ *be sequences of non-negative real* *numbers such that* $a_{n+1} \leq (1+b_n)a_n + c_n$, $\forall n \geq 1$. *If* $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$, *then* $\lim_{n\to\infty} an$ *exists.*

Lemma 2.1.13 *[48] Let* x_n *and* y_n *be two sequences of a uniformly convex hyperbolic space* $(X, d, \mathcal{H})$ *such that, for* $\mathcal{R} \in [0, \infty)$,$\limsup_{n\to\infty} d(x_n, a) \leq \mathcal{R}$, $\limsup_{n\to\infty} d(y_n, a)$ $\leq \mathcal{R}$ *and* $\lim_{n\to\infty} d(\mathcal{H}(x_n, y_n, \alpha_n)) = \mathcal{R}$ *where* $\alpha_n \in [a, b]$ *with* $0 < a < b < 1$, *then we have,* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

CHAPTER III

MAIN RESULTS

3.1 Novel iterative techniques for mixed type for asymptotically nonexpansive mappings in hyperbolic spaces

In this section, we introduce and study the strong convergence of the novel Noor and SP iteration schemes for mixed type asymptotically nonexpansive mappings in the setting of uniformly convex hyperbolic spaces.

3.1.1 A novel Noor iterative technique for mixed type asymptotically nonexpansive mappings in hyperbolic spaces

Let $\mathcal{K}$ be a nonempty closed convex subset of a real uniformly convex hyperbolic space $(X, d, \mathcal{H})$ and $\mathcal{P} : X \to \mathcal{K}$ be a nonexpansive retraction of X onto $\mathcal{K}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to X$ be three asymptotically nonexpansive nonself-mappings. For an arbitrary $x_1 \in \mathcal{K}$, we suggest the following novel Noor iterative scheme for mixed type asypmtotically nonexpansive mappings

$$\begin{aligned} z_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n)), \\ y_n &= \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, \beta_n)), \\ x_{n+1} &= \mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, T_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \gamma_n)), \end{aligned} \tag{3.1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in [0,1).

Lemma 3.1.1 *Let $(X, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of X. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 :$*

$\mathcal{K} \to X$ be three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}$, $\{l_n^{(3)}\} \subset [1, \infty)$ such that, $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega = F(\mathcal{S}_1) \bigcap F(\mathcal{S}_2) \bigcap F(\mathcal{S}_3) \bigcap F(\mathcal{T}_1) \bigcap F(\mathcal{T}_2) \bigcap F(\mathcal{T}_3) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (3.1.1) where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$. Then $\lim_{n\to\infty} d(x_n, v)$ exists for any $v \in \Omega$.

Proof. Using (3.1.1) and setting $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$, we have

$$\begin{aligned} d(z_n, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, T_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n)), v) \\ &\leq d(\mathcal{H}(\mathcal{S}_1^n x_n, T_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n), v) \\ &\leq (1 - \alpha_n) d(\mathcal{S}_1^n x_n, v) + \alpha_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, v) \\ &\leq (1 - \alpha_n) h_n d(x_n, v) + \alpha_n h_n d(x_n, v) \\ &= h_n d(x_n, v) \end{aligned} \tag{3.1.2}$$

and

$$\begin{aligned} d(y_n, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, \beta_n)), v) \\ &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n))), \beta_n)), v) \\ &\leq d(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n)), \beta_n), v) \\ &\leq (1 - \beta_n) d(\mathcal{S}_2^n x_n, v) + \beta_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n)), v) \\ &\leq (1 - \beta_n) h_n^2 d(x_n, v) + \beta_n h_n^2 d(x_n, v) \\ &= h_n^2 d(x_n, v). \end{aligned} \tag{3.1.3}$$

Also,

$$\begin{aligned} d(x_{n+1}, v) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \gamma_n)), v) \\ &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, T_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n x_n, \\ &\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, \alpha_n))), \beta_n))), \gamma_n)), v) \end{aligned}$$

$$
\begin{aligned}
&\leq d(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}(\mathcal{H}(\mathcal{S}_1^n x_n, \\
&\quad \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \beta_n)), \gamma_n), v) \\
&\leq (1-\gamma_n)d(\mathcal{S}_3^n x_n, v) + \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} \\
&\quad (\mathcal{H}(\mathcal{S}_1^n x_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n)), \beta_n)), v) \\
&\leq (1-\gamma_n)h_n^3 d(x_n, v) + \gamma_n h_n^3 d(x_n, v) \\
&= (1 + (h_n^3 - 1))d(x_n, v). \qquad (3.1.4)
\end{aligned}
$$

By the hypothesis, $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$. Therefore, $\sum_{n=1}^{\infty}(h_n^3 - 1) < \infty$. Using Lemma 2.1.12, $\lim_{n\to\infty} d(x_n, v)$ exists. □

Lemma 3.1.2 *Let $(X, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of X. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to X$ be three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}$, $\{l_n^{(2)}\}, \{l_n^{(3)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega = F(\mathcal{S}_1)\bigcap F(\mathcal{S}_2)\bigcap F(\mathcal{S}_3)\bigcap F(\mathcal{T}_1)\bigcap F(\mathcal{T}_2)\bigcap F(\mathcal{T}_3) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (3.1.1) and the following conditions hold:*

(i) $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1)$,

(ii) $d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in \mathcal{K}$ and $i = 1, 2, 3$.

Then $\lim_{n\to\infty} d(x_n, S_i x_n) = \lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i = 1, 2, 3$.

Proof. For any given $v \in \Omega$, $\lim_{n\to\infty} d(x_n, q)$ exists, by Lemma 3.1.1.
Taking $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$. Suppose that $\lim_{n\to\infty}(x_n, v) = c$. By (3.1.4) and $\sum_{n=1}^{\infty}(h_n^3 - 1) < \infty$, we have

$$\lim_{n\to\infty} d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}y_n, \gamma_n)), v) = c \qquad (3.1.5)$$

and

$$\limsup_{n\to\infty} d(\mathcal{S}_3^n x_n, v) \leq \limsup_{n\to\infty} h_n d(x_n, v) = c. \tag{3.1.6}$$

Taking lim sup on both sides of (3.1.3) we obtain,

$$\limsup_{n\to\infty} d(y_n, v) \leq c,$$

and so we have,

$$\limsup_{n\to\infty} d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, v) \leq \limsup_{n\to\infty} d(y_n, v) \leq c. \tag{3.1.7}$$

Using (3.1.5), (3.1.6) and (3.1.7), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) = 0. \tag{3.1.8}$$

By the condition (ii) , we have

$$d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) \leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n). \tag{3.1.9}$$

It follows from (3.1.8) and (3.1.9) that

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) = 0. \tag{3.1.10}$$

In additon,

$$\begin{aligned} d(x_n, v) &\leq d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, v) \\ &\leq d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + h_n d(y_n, v). \end{aligned} \tag{3.1.11}$$

In inequality (3.1.11), taking infimum on both sides and applying (3.1.10), we obtain,

$$\liminf_{n\to\infty} d(y_n, v) \geq c.$$

Since $\limsup_{n\to\infty} d(y_n, v) \leq c$. Therefore, $\lim_{n\to\infty} d(y_n, v) = c$. Using the arguments in (3.1.3) and by $\sum_{n=1}^{\infty}(h_n^{(2)} - 1) < \infty$, we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, \beta_n), v) = c. \tag{3.1.12}$$

In additon,

$$\limsup_{n\to\infty} d(\mathcal{S}_2^n x_n, v) \leq \limsup_{n\to\infty} h_n d(x_n, v) = c. \tag{3.1.13}$$

Taking lim sup on both sides of (3.1.2), we have

$$\limsup_{n\to\infty} d(z_n, v) \leq c. \tag{3.1.14}$$

Using (3.1.14), we have

$$\limsup_{n\to\infty} d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, v) \leq \limsup_{n\to\infty} h_n d(z_n, v) \leq c. \tag{3.1.15}$$

Applying by Lemma 2.1.13, using (3.1.12), (3.1.13) and (3.1.15), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) = 0. \tag{3.1.16}$$

From condition (ii), we get

$$d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) \leq d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n). \tag{3.1.17}$$

It follows from (3.1.16) and (3.1.17) that

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) = 0. \tag{3.1.18}$$

In addition,

$$\begin{aligned} d(x_n, v) &\leq d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, v) \\ &\leq d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + h_n d(z_n, v). \end{aligned} \tag{3.1.19}$$

In the inequality (3.1.19), taking infimum on both sides and applying (3.1.18), we obtain $\liminf_{n\to\infty} d(z_n, v) \geq c$. Since $\limsup_{n\to\infty} d(z_n, v) \leq c$. Therefore, $\lim_{n\to\infty} d(z_n, v) = c$. Using the arguments in (3.1.2) and by $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ we have,

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \alpha_n), v) = c. \tag{3.1.20}$$

In addition,

$$\limsup_{n\to\infty} d(\mathcal{S}_1^n x_n, v) \leq \limsup_{n\to\infty} h_n d(x_n, v) = c \tag{3.1.21}$$

and

$$\limsup_{n\to\infty} d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, v) \leq \limsup_{n\to\infty} h_n d(x_n, v) = c. \tag{3.1.22}$$

Applying by Lemma 2.1.13, using (3.1.20), (3.1.21) and (3.1.22), again we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) = 0. \tag{3.1.23}$$

From condition (ii), we get

$$d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) \leq d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n). \tag{3.1.24}$$

It follows from (3.1.23) and (3.1.24) that

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) = 0. \tag{3.1.25}$$

Using (3.1.1), we have

$$\begin{aligned} d(z_n, \mathcal{S}_1^n x_n) &\leq (1-\alpha_n) d(\mathcal{S}_1^n x_n, \mathcal{S}_1^n x_n) + \alpha_n d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) \\ &= \alpha_n d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n). \end{aligned}$$

It follows from (3.1.23) that

$$\lim_{n\to\infty} d(z_n, \mathcal{S}_1^n x_n) = 0. \tag{3.1.26}$$

Since

$$d(z_n, x_n) \leq d(z_n, \mathcal{S}_1^n x_n) + d(\mathcal{S}_1^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n) + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} x_n, x_n).$$

It follows from (3.1.23), (3.1.25) and (3.1.26) that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.1.27}$$

In addition,

$$d(x_n, \mathcal{S}_1^n x_n) \leq d(x_n, z_n) + d(z_n, \mathcal{S}_1^n x_n).$$

Following from (3.1.26) and (3.1.27), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1^n x_n) = 0. \tag{3.1.28}$$

From (3.1.1), we have

$$d(y_n, \mathcal{S}_2^n x_n) \leq \beta_n d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n). \tag{3.1.29}$$

Following from (3.1.16) and (3.1.29), we have

$$\lim_{n\to\infty} d(y_n, \mathcal{S}_2^n x_n) = 0. \tag{3.1.30}$$

Furthermore,

$$d(y_n, x_n) \leq d(y_n, \mathcal{S}_2^n x_n) + d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} z_n, x_n),$$

by using (3.1.16), (3.1.18) and (3.1.30), we have

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{3.1.31}$$

Since

$$d(x_n, \mathcal{S}_2^n x_n) \leq d(x_n, y_n) + d(y_n, \mathcal{S}_2^n x_n).$$

Using (3.1.30) and (3.1.31), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_2^n x_n) = 0. \tag{3.1.32}$$

Since

$$\begin{aligned} d(x_{n+1}, \mathcal{S}_3^n x_n) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \gamma_n)), \mathcal{S}_3^n x_n) \\ &\leq (1-\gamma_n) d(\mathcal{S}_3^n x_n, \mathcal{S}_3^n x_n) + \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \mathcal{S}_3^n x_n) \\ &\leq \gamma_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \mathcal{S}_3^n x_n). \end{aligned}$$

Using (3.1.8), we have

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{S}_3^n x_n) = 0. \tag{3.1.33}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} x_n) &\leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n, \\ &\quad \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} x_n) \\ &\leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} y_n) + h_n d(y_n, x_n). \end{aligned}$$

It follows from (3.1.8) and (3.1.31) that

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) = 0. \tag{3.1.34}$$

By condition (ii), we know that

$$d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) \leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n).$$

Using (3.1.34), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) = 0. \tag{3.1.35}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) &\leq d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, \\ &\quad \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \\ &\leq d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + h_n d(z_n, x_n). \end{aligned}$$

Using (3.1.16) and (3.1.27), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) = 0. \tag{3.1.36}$$

Again by condition (ii), using (3.1.36), we also have

$$\begin{aligned} d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) &\leq d(\mathcal{S}_2^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \\ &\to 0 \quad (as \;\; n \to \infty). \end{aligned} \tag{3.1.37}$$

Using (3.1.27), (3.1.33) and (3.1.34), we have

$$d(x_{n+1}, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}z_n) \leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n)$$

$$
\begin{aligned}
&\quad +d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}z_n) \\
&\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) + h_n d(x_n, z_n) \\
&\to 0 \quad (as \;\; n \to \infty). \qquad (3.1.38)
\end{aligned}
$$

Since

$$
d(\mathcal{S}_3^n x_n, x_n) \leq d(\mathcal{S}_3^n x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n) + d(x_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}x_n).
$$

Using (3.1.34) and (3.1.45), we have

$$
\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, x_n) = 0. \qquad (3.1.39)
$$

Since

$$
d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \leq d(\mathcal{S}_3^n x_n, x_n) + d(x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n).
$$

It follows from (3.1.37) and (3.1.39) that

$$
\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) = 0. \qquad (3.1.40)
$$

In addition,

$$
\begin{aligned}
d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) &\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) \\
&\quad +d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) \\
&\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}x_n) + h_n d(x_n, z_n).
\end{aligned}
$$

Using (3.1.27), (3.1.33) and (3.1.40), we have

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) = 0. \tag{3.1.41}$$

Since

$$d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) \leq d(\mathcal{S}_3^n x_n, x_n) + d(x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n).$$

Using (3.1.25) and (3.1.39), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) = 0. \tag{3.1.42}$$

Moreover, we have

$$\begin{aligned} d(x_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) &\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) \\ &\quad + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) \\ &\leq d(x_{n+1}, \mathcal{S}_3^n x_n) + d(\mathcal{S}_3^n x_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}x_n) + h_n d(x_n, z_n). \end{aligned}$$

It follows from (3.1.27), (3.1.33) and (3.1.42) that

$$\lim_{n\to\infty} d(x_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}z_n) = 0. \tag{3.1.43}$$

Again, since $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, x_n \in \mathcal{K}$ for $i = 1, 2, 3$ and $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$ are three asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) &= d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, \mathcal{T}_i(\mathcal{P}x_n)) \\ &\leq \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, \mathcal{P}x_n) \\ &\leq \max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-2}z_{n-1}, x_n). \quad (3.1.44) \end{aligned}$$

For $i = 1, 2, 3$, using (3.1.38), (3.1.41) and(3.1.43) in (3.1.44), we have

$$\lim_{n\to\infty} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) = 0. \tag{3.1.45}$$

Since

$$d(x_{n+1}, z_n) \leq d(x_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}z_n, x_n) + d(x_n, z_n),$$

from (3.1.18), (3.1.27) and (3.1.41), we have

$$\lim_{n\to\infty} d(x_{n+1}, z_n) = 0. \tag{3.1.46}$$

In addition, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, \mathcal{T}_i x_n) &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}) \\ &\quad + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n) \\ &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + \max\{\sup_{n\geq 1} l_n^{(1)}, \sup_{n\geq 1} l_n^{(2)}, \sup_{n\geq 1} l_n^{(3)}\} d(x_n, z_{n-1}) \\ &\quad + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}z_{n-1}, \mathcal{T}_i x_n). \end{aligned}$$

Thus, it follows from (3.1.25), (3.1.35), (3.1.37), (3.1.45) and (3.1.46), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_3 x_n) = 0.$$

The first part of the theorem is hence proved. We prove the next part of the theorem, ie.,

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

In fact, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, \mathcal{S}_i x_n) &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{S}_i x_n) \\ &\leq d(x_n, \mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n) + d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}x_n, \mathcal{S}_i^n x_n). \end{aligned}$$

Thus, it follows from (3.1.23), (3.1.25), (3.1.34), (3.1.35), (3.1.36) and (3.1.37) that

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_1 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_2 x_n) = \lim_{n\to\infty} d(x_n, \mathcal{S}_3 x_n) = 0.$$

The proof is completed. □

Let $\{a_n\}$ be a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and ϑ exist with $\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^{\vartheta}} = \lambda$, then $\{a_n\}$ converges to a of order ϑ, with asymptotic error constant λ. If $\vartheta = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\vartheta = 2$, the sequence is quadratically convergent (see [18]).

The following example presents the condition (ii) in Lemma 3.1.2.

Example 3.1.3 [57] *Let X be a real line with metric $d(x, y) = |x - y|$ and $\mathcal{K} = [-1, 1]$. Define $\mathcal{H} : X \times X \times [0, 1] \to X$ by $\mathcal{H}(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Then $(X, d, \mathcal{H})$ is complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\mathcal{K}$ is a nonempty closed convex subset of X. Define two mappings $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ by*

$$\mathcal{T}x = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0) \end{cases}$$

and

$$\mathcal{S}x = \begin{cases} x, & \text{if } x \in [0, 1], \\ -x, & \text{if } x \in [-1, 0). \end{cases}$$

Clearly, $F(\mathcal{T}) = \{0\}$ *and* $F(\mathcal{S}) = \{x \in \mathcal{K}; 0 \leq x \leq 1\}$*. Now, we show that* $\mathcal{T}$ *is nonexpansive. In fact, if* $x, y \in [0,1]$ *or* $x, y \in [-1, 0)$*, then*

$$d(\mathcal{T}x, \mathcal{T}y) = |\mathcal{T}x - \mathcal{T}y| = 2|\sin\frac{x}{2} - \sin\frac{y}{2}| \leq |x - y| = d(x, y).$$

If $x \in [0,1]$ *and* $y \in [-1, 0)$ *or* $x \in [-1, 0)$ *and* $y \in [0, 1]$*, then*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\sin\frac{x}{2} + \sin\frac{y}{2}| \\ &= 4|\sin\frac{x+y}{4}\cos\frac{x-y}{4}| \\ &\leq |x + y| \\ &\leq |x - y| \\ &= d(x, y). \end{aligned}$$

That is, $\mathcal{T}$ *is nonexpansive. It follows that* $\mathcal{T}$ *is an asymptotically nonexpansive mapping with* $k_n = 1$ *for each* $n \geq 1$*. Similarly, we can show that* $\mathcal{S}$ *is an asymptotically nonexpansive mapping with* $l_n = 1$ *for each* $n \geq 1$*. Next, to show that* $\mathcal{S}$ *and* $\mathcal{T}$ *satisfy the condition (ii) in Lemma 3.1.2, we have to consider the following cases:*

Case 1. Let $x, y \in [0, 1]$*. It follows that*

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 2. Let $x, y \in [-1, 0)$*. It follows that*

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| \leq |-x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 3. Let $x \in [-1, 0)$ *and* $y \in [0, 1]$. *It follows that*

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x + 2\sin\frac{y}{2}| \leq | - x + 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Case 4. Let $x \in [0, 1]$ *and* $y \in [-1, 0]$. *It follows that*

$$d(x, \mathcal{T}y) = |x - \mathcal{T}y| = |x - 2\sin\frac{y}{2}| = |\mathcal{S}x - \mathcal{T}y| = d(\mathcal{S}x, \mathcal{T}y).$$

Hence the condition (ii) in Lemma 3.1.2 is satisfied. In addition, let $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$ *and* $\gamma_n = \frac{n}{4n+1}$, $\forall n \geq 1$. *Consequently, the conditions of Lemma 3.1.2 are fulfilled. Thus, the convergence of the sequence* $\{x_n\}$ *generated by* (3.1.1) *to a point* $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ *can be recived.* $\square$

Now, we present some numerical examples to illustrate the convergence and efficiency of the proposed algorithms. We choose $x_1 = 1$ and run our process within 100 iterations. All codes were written in Matlab 2022a. We obtain the iteration steps and its amplification factor of the proposed algorithms as shown in Table 1. For convenience, we call the iteration (3.1.1) the proposed iteration process.

Table 3.1.1: Numerical experiment of the proposed method for Example 3.1.3

The Proposed Iteration Process		
Iteration Number (n)	$\lvert x_n \rvert$	$\frac{\lvert x_{n+1} \rvert}{\lvert x_n \rvert}$
1	1.0000e+00	1.8283e-01
2	1.8283e-01	1.1064e-01
3	2.0229e-02	7.6918e-02
4	1.5559e-03	5.8824e-02
5	9.1526e-05	4.7619e-02
⋮	⋮	⋮
10	2.6686e-15	2.4390e-02
⋮	⋮	⋮
20	1.7026e-33	1.2346e-02
⋮	⋮	⋮
40	2.0079e-75	6.2112e-03
⋮	⋮	⋮
60	1.0911e-121	4.1494e-03
⋮	⋮	⋮
80	8.0992e-171	3.1153e-03
⋮	⋮	⋮
100	4.2888e-222	2.4938e-03

Table 1 show that the proposed method converges to zero. It can be concluded that the proposed method is linearly convergent and its amplification factor less than 0.003.

Next, we can prove a strong convergence theorem.

Theorem 3.1.4 *Let* $\mathcal{K}$, X, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$ *satisfy the hypotheses of Lemma 3.1.2. Suppose that* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$ *and* $\mathcal{S}_i$, $\mathcal{T}_i$ *for all* $i = 1,2,3$ *satisfy the condition (ii) in Lemma 3.1.2. If there is a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$ *for all* $r \in (0,\infty)$ *such that*

$$f(d(x,\Omega)) \leq d(x,\mathcal{S}_1 x) + d(x,\mathcal{S}_2 x) + d(x,\mathcal{S}_3 x) + d(x,\mathcal{T}_1 x) + d(x,\mathcal{T}_2 x) + d(x,\mathcal{T}_3 x)$$

for all $x \in \mathcal{K}$, *where* $d(x,\Omega) = \inf\{d(x,v) : v \in \Omega\}$. *Then the sequence* $\{x_n\}$ *defined by algorithm* (3.1.2) *converges strongly to a common fixed point of* $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$.

Proof. *From Lemma 3.1.2, we have* $\lim_{n\to\infty} d(x_n, \mathcal{S}_i x_n) = 0 = \lim_{n\to\infty} d(x_n, \mathcal{T}_i x_n)$ *for* $i = 1, 2, 3$. *It follows from the hypothesis that*

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n,\Omega)) &\leq \lim_{n\to\infty} (d(x_n,\mathcal{S}_1x_n) + d(x_n,\mathcal{S}_2x_n) + d(x_n,\mathcal{S}_3x_n) \\ &\quad + d(x_n,\mathcal{T}_1x_n) + d(x_n,\mathcal{T}_2x_n) + d(x_n,\mathcal{T}_3x_n)) \\ &= 0.\end{aligned}$$

Thus $\lim_{n\to\infty} f(d(x_n,\Omega)) = 0$. *Since* $f : [0,\infty) \to [0,\infty)$ *is a nondecreasing function satisfying* $f(0) = 0, f(r) > 0$ *for all* $r \in (0,\infty)$. *By Lemma 3.1.1, we obtain that* $\lim_{n\to\infty} d(x_n,\Omega)$ *exists. This implies that* $\lim_{n\to\infty} d(x_n,\Omega) = 0$. *Next, we show that* $\{x_n\}$ *is a Cauchy sequence in* $\mathcal{K}$. *Using* (3.1.4), *we have*

$$d(x_{n+1}, v) \leq (1 + (h_n^3 - 1))d(x_n, v)$$

for each $n \geq 1$, *where* $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$ *and* $v \in \Omega$. *For any* $m, n > n \geq 1$, *we have*

$$\begin{aligned}d(x_m, v) &\leq (1 + (h_{m-1}^3 - 1))d(x_{m-1}, v) \\ &\leq e^{h_{m-1}^3 - 1} d(x_{m-1}, v) \\ &\leq e^{h_{m-1}^3 - 1} e^{h_{m-2}^3 - 1} d(x_{m-2}, v) \\ &\vdots \\ &\leq e^{\sum_{i=n}^{m-1}} h_i^3 - 1 d(x_n, v) \\ &\leq M d(x_n, v),\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^3-1)}$. *So, for any* $v \in \Omega$, *we have*

$$d(x_n, x_m) \leq d(x_n, v) + d(x_m, v) \leq (1+M)d(x_n, v).$$

Taking the infimum over all $v \in \Omega$, *we have*

$$d(x_n, x_m) \leq (1+M)d(x_n, \Omega).$$

Thus it follows from $\lim_{n\to\infty} d(x_n, \Omega) = 0$ *that* $\{x_n\}$ *is a Cauchy sequence. Since* $\mathcal{K}$ *is a closed subset in a complete hyperbolic space* X, *the sequence* $\{x_n\}$ *converges strongly to some* $v^* \in \mathcal{K}$. *It is easy to prove that* $F(\mathcal{S}_1),\ F(\mathcal{S}_2),\ F(\mathcal{S}_3),\ F(\mathcal{T}_1),\ F(\mathcal{T}_2)$ *and* $F(\mathcal{T}_3)$ *are all closed, that is,* Ω *is closed subset of* $\mathcal{K}$. *Since* $\lim_{n\to\infty} d(x_n, \Omega) = 0$ *gives that* $d(v^*, \Omega) = 0$, *we have* $v^* \in \Omega$. *The proof is completed.* □

Theorem 3.1.5 *Considering the assumption in Lemma 3.1.2 and if one of* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *and* $\mathcal{T}_3$ *is completely continuous after that the sequence* $\{x_n\}$ *defined by 3.1.1 converges strongly to a point in* Ω.

Proof. *Let* $\mathcal{S}_1$ *be completely continuous. By Lemma 3.1.1,* $\{x_n\}$ *is bounded. This mean, there is a subsequence* $\{\mathcal{S}_1 x_{n_j}\}$ *of* $\{\mathcal{S}_1 x_n\}$ *such that* $\{\mathcal{S}_1 x_{n_j}\}$ *converges strongly to some* $v^* \in \mathcal{K}$. *Moreover, by Lemma 3.1.2, we have*

$$\lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_1 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_2 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{S}_3 x_n) = 0 \text{ and}$$

$$\lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_1 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_2 x_n) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_3 x_n) = 0,$$

which implies that,

$$\begin{aligned} d(x_{n_j}, v^*) &\leq d(x_{n_j}, \mathcal{S}_1 x_{n_j}) + d(\mathcal{S}_1 x_{n_j}, v^*) \\ &\to 0 \quad (as\ j \to \infty). \end{aligned}$$

Hence $\mathcal{S}_1 x_{n_j} \to v^* \in K$. *Consequently,*

$$d(v^*, \mathcal{S}_i v^*) = \lim_{n\to\infty} d(x_{n_j}, \mathcal{S}_i x_{n_j}) = 0.$$

Since $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *and* $\mathcal{T}_3$ *are continuous, for* $i = 1, 2, 3$. *By Lemma 3.1.2, so we have*

$$d(v^*, \mathcal{T}_i v^*) = \lim_{j\to\infty} d(x_{n_j}, \mathcal{T}_i x_{n_j}) = 0.$$

This implies that $v^* \in F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. *From Lemma 3.1.1, we have* $\lim_{n\to\infty} d(x_n, v^*)$ *exists and so* $\lim_{n\to\infty} d(x_n, v^*) = 0$. *Thus* $\{x_n\}$ *converges strongly to a common fixed point of* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *and* $\mathcal{T}_3$ *The proof is completed.* □

3.1.2 Mixed type SP-iteration for asymptotically nonexpansive mappings in hyperbolic spaces

In this section, we suggest a mixed type SP-iteration for three asymptotically nonexpansive self and nonself mappings in the setting of uniformly convex hyperbolic spaces.

Let $(X, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of X. Suppose that $\mathcal{P} : X \to \mathcal{K}$ is a nonexpansive retraction, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings, and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to X$ are three asymptotically nonexpansive nonself-mappings. The set of common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ denoted by $\Omega := F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. The iteration procedure that follows is a translation of the SP-iteration presented in [70] from Banach spaces to hyperbolic spaces:

$$\begin{cases} u_1 \in \mathcal{K}, \\ w_n = \mathcal{P}(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, \alpha_n)), \\ v_n = \mathcal{P}(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n)), \\ u_{n+1} = \mathcal{P}(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n)), \quad n \leq 1, \end{cases} \tag{3.1.47}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$.

We now prove a strong convergence theorem for X, using the iterative scheme given in (3.1.47). The following lemmas are needed.

Lemma 3.1.6 *Let $\emptyset \neq \mathcal{K}$ be a closed convex subset of a uniformly convex hyperbolic space $(X, d, \mathcal{H})$. Suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}$, $\{k_n^{(2)}\}$, $\{k_n^{(3)}\} \subset [1, \infty)$, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to X$ are three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\}$ $\subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, respectively, and $\Omega \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[0, 1)$. From $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ using (3.1.47). Then $\lim_{n\to\infty} d(u_n, p)$ exists, $\forall p \in \Omega$.*

Proof. Let $p \in \Omega$ and setting $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$. From (3.1.47), we have

$$\begin{aligned} d(w_n, p) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, \alpha_n)), p) \\ &\leq d(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(\mathcal{S}_3^n u_n, p) + \alpha_n d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, p) \\ &\leq (1 - \alpha_n) h_n d(u_n, p) + \alpha_n h_n d(u_n, p) \\ &= h_n d(u_n, p) \end{aligned} \tag{3.1.48}$$

and

$$\begin{aligned} d(v_n, p) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n)), p) \\ &\le d(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n), p) \\ &\le (1-\beta_n) d(\mathcal{S}_2^n w_n, p) + \beta_n d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, p) \\ &\le (1-\beta_n) h_n d(w_n, p) + \beta_n h_n d(w_n, p) \\ &= h_n d(w_n, p) \\ &\le h_n^2 d(u_n, p). \end{aligned} \tag{3.1.49}$$

Using (3.1.49), we have

$$\begin{aligned} d(u_{n+1}, p) &= d(\mathcal{P}(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n)), p) \\ &\le d(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n), p) \\ &\le (1-\gamma_n) d(\mathcal{S}_1^n v_n, p) + \gamma_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, p) \\ &\le (1-\gamma_n) h_n d(v_n, p) + \gamma_n h_n d(v_n, p) \\ &= h_n d(v_n, p) \\ &\le h_n^3 d(u_n, p) \\ &= (1 + (h_n^3 - 1)) d(u_n, p). \end{aligned} \tag{3.1.50}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, we have $\sum_{n=1}^{\infty}(h_n^{(3)} - 1) < \infty$. Using Lemma 2.1.12, $\lim_{n\to\infty} d(u_n, p)$ exists. □

Lemma 3.1.7 *Let $\emptyset \neq \mathcal{K}$ be a closed convex subset of a uniformly convex hyperbolic space $(X, d, \mathcal{H})$. Suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ are three asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}$, $\{k_n^{(2)}\}, \{k_n^{(3)}\} \subset [1, \infty)$, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to X$ are three asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\}, \{l_n^{(3)}\}$ $\subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2, 3$, re-*

spectively, and $\Omega \neq \emptyset$*. Assume* $\{u_n\}$ *be a sequence defined by* (3.1.47) *and the following conditions hold:*

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[\varepsilon, 1-\varepsilon]$, $\exists \varepsilon \in (0,1)$,

(ii) $d(u, \mathcal{T}_i v) \leq d(\mathcal{S}_i u, \mathcal{T}_i v)$, $\forall u, v \in \mathcal{K}$, $i = 1, 2, 3$.

Then $\lim_{n\to\infty} d(u_n, \mathcal{S}_i u_n) = \lim_{n\to\infty} d(u_n, \mathcal{T}_i u_n) = 0$ *for* $i = 1, 2, 3$.

Proof. Let $p \in \Omega$ and setting $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$. From Lemma 3.1.6, we have $\lim_{n\to\infty} d(u_n, p)$ exists. Suppose that $\lim_{n\to\infty} d(u_n, p) = c$, letting $n \to \infty$ in (3.1.50), we get

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n), p) = c. \tag{3.1.51}$$

Using (3.1.49), we obtain $d(\mathcal{S}_1^n v_n, p) \leq h_n^3 d(u_n, p)$. Using the lim sup on both sides of this inequality, we get

$$\limsup_{n\to\infty} d(S_1^n v_n, p) \leq c. \tag{3.1.52}$$

Taking the lim sup in (3.1.49), we get $\limsup_{n\to\infty} d(v_n, p) \leq c$. Thus

$$\limsup_{n\to\infty} d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, p) \leq \limsup_{n\to\infty} h_n d(v_n, p) = c. \tag{3.1.53}$$

By (3.1.51),(3.1.52),(3.1.53) and Lemma 2.1.13, we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) = 0. \tag{3.1.54}$$

Using condition (ii), we have

$$\lim_{n\to\infty} d(v_n, (\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) \leq \lim_{n\to\infty} d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n). \tag{3.1.55}$$

Using (3.1.55), we obtain

$$\lim_{n\to\infty} d(v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) = 0. \tag{3.1.56}$$

From (3.1.50), we obtain

$$\begin{aligned} d(u_{n+1}, p) &\leq d(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \gamma_n), p) \\ &\leq (1-\gamma_n)d(\mathcal{S}_1^n v_n, p) + \gamma_n d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) + \gamma_n d(\mathcal{S}_1^n v_n, p) \\ &= d(\mathcal{S}_1^n v_n, p) + \gamma_n d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \\ &\leq h_n d(v_n, p) + \gamma_n d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n). \end{aligned} \tag{3.1.57}$$

Taking the lim inf into consideration on both sides of the inequality (3.1.57), using (3.1.54), $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $\lim_{n\to\infty} d(u_{n+1}, p) = c$, we have

$$\liminf_{n\to\infty} d(v_n, p) \geq c. \tag{3.1.58}$$

Since $\limsup_{n\to\infty} d(v_n, p) \leq c$, by (3.1.58), we have

$$\lim_{n\to\infty} d(v_n, p) = c.$$

Letting $n \to \infty$ in (3.1.49), we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n), p) = c. \tag{3.1.59}$$

In addition, using (3.1.48), we obtain $d(\mathcal{S}_2^n w_n, p) \leq h_n^2 d(u_n, p)$. Taking the lim sup on both sides of this inequality, we obtain

$$\limsup_{n\to\infty} d(\mathcal{S}_2^n w_n, p) \leq c. \tag{3.1.60}$$

Taking the lim sup in (3.1.48), we get $\limsup_{n\to\infty} d(w_n, p) \le c$. Thus

$$\limsup_{n\to\infty} d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, p) \le \limsup_{n\to\infty} h_n d(w_n, p) = c. \tag{3.1.61}$$

Using Lemma 2.1.13, by (3.1.59), (3.1.60) and (3.1.61), we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) = 0. \tag{3.1.62}$$

Using condition (ii), we have

$$\lim_{n\to\infty} d(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) \le \lim_{n\to\infty} d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n),$$

and thus

$$\lim_{n\to\infty} d(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) = 0. \tag{3.1.63}$$

From (3.1.49), we get

$$\begin{aligned}
d(v_n, p) &\le d(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \beta_n), p) \\
&\le (1-\beta_n) d(S_2^n w_n, p) + \beta_n d(S_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) + \beta_n d(\mathcal{S}_2^n w_n, p) \\
&= d(\mathcal{S}_2^n w_n, p) + \beta_n d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) \\
&\le h_n d(w_n, p) + \beta_n d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n).
\end{aligned} \tag{3.1.64}$$

Taking the lim inf into consideration on both sides of the inequality (3.1.64), using (3.1.62), $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $\lim_{n\to\infty} d(v_n, p) = c$, we have

$$\liminf_{n\to\infty} d(w_n, p) \ge c. \tag{3.1.65}$$

Since $\limsup_{n\to\infty} d(w_n, p) \le \limsup_{n\to\infty} h_n d(u_n, p) \le c$, by (3.1.65), we have

$$\lim_{n\to\infty} d(w_n, p) = c.$$

Letting $n \to \infty$ in the inequality (3.1.48), we get

$$\begin{aligned} c = \lim_{n\to\infty} d(w_n, p) &\leq \lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n), p) \\ &\leq \lim_{n\to\infty} d(u_n, p) = c, \end{aligned}$$

and so

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \alpha_n), p) = c. \tag{3.1.66}$$

Moreover, we obtain

$$\limsup_{n\to\infty} d(\mathcal{S}_3^n u_n, p) \leq \limsup_{n\to\infty} h_n d(u_n, p) = c \tag{3.1.67}$$

and

$$\limsup_{n\to\infty} d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, p) \leq \limsup_{n\to\infty} h_n d(u_n, p) = c. \tag{3.1.68}$$

Following (3.1.66), (3.1.67), (3.1.68) and Lemma 2.1.13, we get

$$\lim_{n\to\infty} d(S_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) = 0. \tag{3.1.69}$$

Next, we show that

$$\lim_{n\to\infty} d(u_n, \mathcal{T}_1 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{T}_2 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{T}_3 u_n) = 0.$$

Indeed, condition (ii) implies

$$d(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) \leq d(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n). \tag{3.1.70}$$

By (3.1.69) and (3.1.70), which implies that

$$\lim_{n\to\infty} d(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) = 0. \tag{3.1.71}$$

Using (3.1.47), we have

$$\begin{aligned} d(w_n, \mathcal{S}_3^n u_n) &\le (1-\alpha_n) d(\mathcal{S}_3^n u_n, \mathcal{S}_3^n u_n) + \alpha_n d(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n) \\ &= \alpha_n d(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n). \end{aligned}$$

Following from (3.1.69),

$$\lim_{n\to\infty} d(w_n, \mathcal{S}_3^n u_n) = 0. \tag{3.1.72}$$

In addition, we have

$$d(w_n, u_n) \le d(w_n, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n) + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1} u_n, u_n). \tag{3.1.73}$$

Using (3.1.69), (3.1.71), (3.1.72) and (3.1.73), we have

$$\lim_{n\to\infty} d(w_n, u_n) = 0. \tag{3.1.74}$$

Furthermore,

$$d(\mathcal{S}_2^n w_n, w_n) \le d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n),$$

by using (3.1.62) and (3.1.63), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n w_n, w_n) = 0. \tag{3.1.75}$$

It follows from (3.1.47), (3.1.63) and (3.1.75) that

$$\begin{aligned} d(v_n, w_n) &= d(\mathcal{H}(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, \beta_n), w_n) \\ &\le (1-\beta_n) d(\mathcal{S}_2^n w_n, w_n) + \beta_n d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.1.76}$$

And then, from (3.1.74) and (3.1.76), we have

$$\begin{aligned} d(v_n, u_n) &\leq d(v_n, w_n) + d(w_n, u_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.77}$$

By the condition (ii), we know that

$$d(u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) \leq d(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n). \tag{3.1.78}$$

Since

$$\begin{aligned} d(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) &\leq d(\mathcal{S}_1^n u_n, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \\ &\quad + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) \\ &\leq h_n d(u_n, v_n) + d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \\ &\quad + h_n d(v_n, u_n). \end{aligned} \tag{3.1.79}$$

Using (3.1.54) and (3.1.77) in (3.1.79), we obtain

$$\lim_{n\to\infty} d(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) = 0. \tag{3.1.80}$$

By using (3.1.78) and (3.1.80), we obtain

$$\lim_{n\to\infty} d(u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) = 0. \tag{3.1.81}$$

From (3.1.63) and (3.1.74), we have

$$\begin{aligned} d(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) &\leq d(u_n, w_n) + d(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) \\ &\quad + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \\ &\leq d(u_n, w_n) + d(w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) + h_n d(w_n, u_n) \end{aligned}$$

$$\to 0 \quad (\text{ as } \ n \to \infty). \tag{3.1.82}$$

Using (3.1.62), (3.1.63) and (3.1.74), we have

$$\begin{aligned} d(\mathcal{S}_2^n u_n, u_n) &\leq d(\mathcal{S}_2^n u_n, \mathcal{S}_2^n w_n) + d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) \\ &+ d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) + d(w_n, u_n) \\ &\leq h_n d(u_n, w_n) + d(\mathcal{S}_2^n w_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n) \\ &+ d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} w_n, w_n) + d(w_n, u_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.83}$$

It follows from (3.1.82) and (3.1.83) that

$$\begin{aligned} d(\mathcal{S}_2^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) &\leq d(\mathcal{S}_2^n u_n, u_n) + d(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.84}$$

Using (3.1.54), we have

$$\begin{aligned} d(u_{n+1}, \mathcal{S}_1^n v_n) &= d(\mathcal{H}(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \gamma_n), \mathcal{S}_1^n v_n) \\ &\leq (1-\gamma_n) d(\mathcal{S}_1^n v_n, \mathcal{S}_1^n v_n) + \gamma_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \mathcal{S}_1^n v_n) \\ &= \gamma_n d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \mathcal{S}_1^n v_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.85}$$

By (3.1.54) and (3.1.85), we have

$$\begin{aligned} d(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) &\leq d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.86}$$

Using (3.1.76) and (3.1.86), we have

$$\begin{aligned} d(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}w_n) &\le d(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) \\ &\quad + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}w_n) \\ &\le d(u_{n+1}, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n) + h_n d(v_n, w_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.1.87}$$

Moreover, from (3.1.80) and (3.1.81), we have

$$\begin{aligned} d(\mathcal{S}_1^n u_n, u_n) &\le d(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n) + d(\mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}u_n, u_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.1.88}$$

Using (3.1.82) and (3.1.88), we have

$$\begin{aligned} d(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) &\le d(\mathcal{S}_1^n u_n, u_n) + d(u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.1.89}$$

It follows from (3.1.77) and (3.1.89) that

$$\begin{aligned} d(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) &\le d(\mathcal{S}_1^n v_n, \mathcal{S}_1^n u_n) + d(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \\ &\le h_n d(v_n, u_n) + d(\mathcal{S}_1^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \\ &\to 0 \quad (\text{ as } n \to \infty). \end{aligned} \tag{3.1.90}$$

Using (3.1.74), (3.1.85) and (3.1.90), we have

$$\begin{aligned} d(u_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) &\le d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) \\ &\quad + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) \\ &\le d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n) + h_n d(u_n, w_n) \end{aligned}$$

$$\to 0 \quad (\text{ as } \ n \to \infty). \tag{3.1.91}$$

In addition, using (3.1.71), (3.1.74), (3.1.77), (3.1.85) and (3.1.88), we obtain

$$\begin{aligned} d(u_{n+1}, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}w_n) &\leq d(u_{n+1}, \mathcal{S}_1^n v_n) + d(\mathcal{S}_1^n v_n, \mathcal{S}_1^n u_n) + d(\mathcal{S}_1^n u_n, u_n) \\ &\quad + d(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) + d(\mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n, \\ &\quad \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}w_n) \\ &\leq d(u_{n+1}, \mathcal{S}_1^n v_n) + h_n d(v_n, u_n) + d(\mathcal{S}_1^n u_n, u_n) \\ &\quad + d(u_n, \mathcal{T}_3(\mathcal{P}\mathcal{T}_3)^{n-1}u_n) + h_n d(u_n, w_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.92}$$

From $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, u_n \in \mathcal{K}$ $(i = 1, 2, 3)$, and $\mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_3$ are three asymptotically nonexpansive nonself-mappings, we get

$$\begin{aligned} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n) &= d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, \mathcal{T}_i(\mathcal{P}u_n)) \\ &\leq max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, \mathcal{P}u_n) \\ &\leq max\{l_1^{(1)}, l_1^{(2)}, l_1^{(3)}\} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-2}w_{n-1}, u_n). \end{aligned} \tag{3.1.93}$$

Using (3.1.87), (3.1.91), (3.1.92) and (3.1.93), for $i = 1, 2, 3$, we obtain

$$\lim_{n \to \infty} d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}, \mathcal{T}_i u_n) = 0. \tag{3.1.94}$$

By using (3.1.63) and (3.1.91), we have

$$\begin{aligned} d(u_{n+1}, w_n) &\leq d(u_{n+1}, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n) + d(\mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}w_n, w_n) \\ &\to 0 \quad (\text{ as } \ n \to \infty). \end{aligned} \tag{3.1.95}$$

Moreover, for $i=1,2,3$, we have

$$\begin{aligned} d(u_n,\mathcal{T}_i u_n) &\le d(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n,(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1}) \\ &+ d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1},\mathcal{T}_i u_n) \\ &\le d(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+max\{\sup_{n\ge 1} l_1^{(1)},\sup_{n\ge 1} l_2^{(2)},\sup_{n\ge 1} l_3^{(3)}\}d(u_n,w_{n-1}) \\ &+ d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}w_{n-1},\mathcal{T}_i u_n). \end{aligned}$$

Therefore, it follows from (3.1.71), (3.1.81), (3.1.82), (3.1.94) and (3.1.95) that

$$\lim_{n\to\infty} d(u_n,\mathcal{T}_1 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{T}_2 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{T}_3 u_n)=0.$$

Lastly, we prove that

$$\lim_{n\to\infty} d(u_n,\mathcal{S}_1 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{S}_2 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{S}_3 u_n)=0.$$

In fact, for $i=1,2,3$, we have

$$\begin{aligned} d(u_n,\mathcal{S}_i u_n) &\le d(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n,\mathcal{S}_i u_n) \\ &\le d(u_n,\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n)+d(\mathcal{T}_i(\mathcal{P}\mathcal{T}_i)^{n-1}u_n,\mathcal{S}_i^n u_n). \end{aligned}$$

So, it follows from (3.1.69), (3.1.71), (3.1.80), (3.1.81), (3.1.82) and (3.1.84) that

$$\lim_{n\to\infty} d(u_n,\mathcal{S}_1 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{S}_2 u_n)=\lim_{n\to\infty} d(u_n,\mathcal{S}_3 u_n)=0.$$

□

Example 3.1.8 [57] *Suppose that $\mathcal{K}=[-1,1]$ is a subset of a real line X with $d(u,v)=|u-v|$ and $\mathcal{H}:X\times X\times[0,1]\to X$ defined by $\mathcal{H}(u,v,\alpha):=\alpha u+(1-\alpha)v$, $\forall u,v\in X$, $\alpha\in[0,1]$. We have that $(X,d,\mathcal{H})$ is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and $\emptyset\neq\mathcal{K}\subseteq X$ is a closed* *and convex. Let* $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ *be two mappings defined by*

$$\mathcal{T}u = \begin{cases} -2\sin\frac{u}{2}, & u \in [0,1], \\ 2\sin\frac{u}{2}, & u \in [-1,0) \end{cases}$$

and

$$\mathcal{S}u = \begin{cases} u, & u \in [0,1], \\ -u, & u \in [-1,0). \end{cases}$$

We have that $F(\mathcal{T}) = \{0\}$ *and* $F(\mathcal{S}) = \{u \in \mathcal{K}; 0 \le u \le 1\}$. *We prove that* $\mathcal{T}$ *is nonexpansive. Indeed, assume that* $u, v \in [0,1]$ *or* $u, v \in [-1,0)$. *Then*

$$d(\mathcal{T}u, \mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v| = 2|\sin\frac{u}{2} - \sin\frac{v}{2}| \le |u - v| = d(u,v).$$

Assume that $u \in [0,1]$, $v \in [-1,0)$ *or* $u \in [-1,0)$, $v \in [0,1]$. *Then*

$$\begin{aligned} d(\mathcal{T}u, \mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v| &= 2|\sin\frac{u}{2} + \sin\frac{v}{2}| \\ &= 4|\sin\frac{u+v}{4}\cos\frac{u-v}{4}| \\ &\le |u+v| \\ &\le |u-v| \\ &= d(u,v). \end{aligned}$$

Hence $\mathcal{T}$ *is nonexpansive. That is,* $\mathcal{T}$ *is an asymptotically nonexpansive mapping with* $k_n = 1$, $\forall n \ge 1$. *Similarly, we can prove that* $\mathcal{S}$ *is an asymptotically nonexpansive mapping with* $l_n = 1$, $\forall n \ge 1$. *Then, to demonstrate that* $\mathcal{S}$ *and* $\mathcal{T}$ *fulfill condition (ii) of Lemma 3.1.7, we must examine the following cases:*

Case (i). *Let* $u, v \in [0,1]$. *We have*

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u + 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

Case (ii). Let $u, v \in [-1, 0)$*. We have*

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u - 2\sin\frac{v}{2}| \leq |-u - 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

Case (iii). Let $u \in [-1, 0)$ *and* $v \in [0, 1]$*. We have*

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u + 2\sin\frac{v}{2}| \leq |-u + 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

Case (iv). Let $u \in [0, 1]$ *and* $v \in [-1, 0]$*. We have*

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = |u - 2\sin\frac{v}{2}| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

It follows that the condition (ii) in Lemma 3.1.7 is satisfied. Moreover, we take $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$ *and* $\gamma_n = \frac{n}{4n+1}$, $\forall n \geq 1$*. We have that the conditions of Lemma 3.1.7 are fulfilled. Consequently, a convergence of the sequence* $\{u_n\}$ *produced by* (3.1.47) *to the point* $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ *can be obtained.* □

Now, we provide some numerical examples to illustrate the convergence behavior of iteration (1.0.7) comparing with iteration (3.1.47). All program computation are performed on an Hp Laptop Intel(R) Core(TM) i7-1165G7, 16.00 GB RAM. We choose the starting point at $u_1 = 1$ and the stop criterion is defined by $||u_n - 0|| < 10^{-15}$. The convergence performance of both iteration are shown in the following Table 1 and Figure 1.

Under the same condition settings shown in Example 3.1.8, by Table 1 and Figure 1, our proposed iteration (3.1.47) has a better performance in both the time taken by CPU-runtime to reach the convergence and the number of iterations when comparing with iteration (1.0.7).

Table 3.1.2: Computational result for all setting in Example 3.1.8

	Iteration (1.0.7)	Iteration (3.1.47)
No of Iter.	26	10
CPU time (sec)	0.0035	0.0027

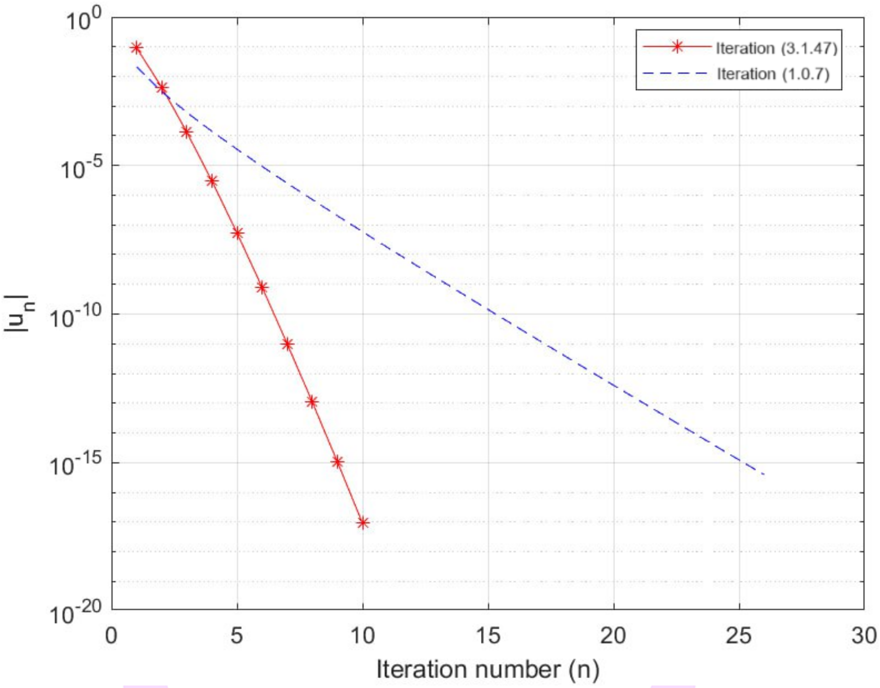

Figure 3.1.1: The value of $\{u_n\}$ generated by iteration (1.0.7) and iteration (3.1.47)

The next step is to prove strong convergence theorems.

Theorem 3.1.9 *Let* $\mathcal{K}$, X, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ *satisfy the hypotheses of Lemma 3.1.7,* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[\varepsilon, 1-\varepsilon]$, $\exists\varepsilon \in (0,1)$, *and* $\mathcal{S}_i$, $\mathcal{T}_i$ *for any* $i = 1, 2, 3$ *satisfy the condition (ii) in Lemma 3.1.7. Suppose that there is a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$, $\forall r \in (0,\infty)$ *such that*

$$f(d(u,\Omega)) \leq d(u,\mathcal{S}_1u) + d(u,\mathcal{S}_2u) + d(u,\mathcal{S}_3u) + d(u,\mathcal{T}_1u) + d(u,\mathcal{T}_2u) + d(u,\mathcal{T}_3u),$$

$\forall u \in \mathcal{K}$, *where* $d(u,\Omega) = \inf\{d(u,p) : p \in \Omega\}$. *Then the sequence* $\{u_n\}$ *defined by*

(3.1.47) *converges strongly to a common fixed point of* $\mathcal{S}_1,\ \mathcal{S}_2,\ \mathcal{S}_3,\ \mathcal{T}_1,\ \mathcal{T}_2$ *and* $\mathcal{T}_3$.

Proof. From Lemma 3.1.7, we have $\lim_{n\to\infty} d(u_n, \mathcal{S}_i u_n) = 0 = \lim_{n\to\infty} d(u_n, \mathcal{T}_i u_n)$ $(i = 1, 2, 3)$. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(u_n, \Omega)) \leq \lim_{n\to\infty} &(d(u_n, \mathcal{S}_1 u_n) + d(u_n, \mathcal{S}_2 u_n) + d(u_n, \mathcal{S}_3 u_n)\\ &+ d(u_n, \mathcal{T}_1 u_n) + d(u_n, \mathcal{T}_2 u_n) + d(u_n, \mathcal{T}_3 u_n))\\ &= 0.\end{aligned}$$

Hence $\lim_{n\to\infty} f(d(u_n, \Omega)) = 0$. From $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0,\ \forall r \in (0, \infty)$. Using Lemma 3.1.6, we have $\lim_{n\to\infty} d(u_n, \Omega)$ exists. It follows that $\lim_{n\to\infty} d(u_n, \Omega) = 0$. Next, we prove that $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}$. Using (3.1.50), we have

$$d(u_{n+1}, p) \leq (1 + (h_n^3 - 1))d(u_n, p),$$

$\forall n \geq 1$, where $h_n = max\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\}$ and $p \in \Omega$. For all $m, n > n \geq 1$, we obtain

$$\begin{aligned}d(u_m, p) &\leq (1 + (h_{m-1}^3 - 1))d(u_{m-1}, p)\\ &\leq e^{h_{m-1}^3 - 1} d(u_{m-1}, p)\\ &\leq e^{h_{m-1}^3 - 1} e^{h_{m-2}^3 - 1} d(u_{m-2}, p)\\ &\ \vdots\\ &\leq e^{\sum_{i=n}^{m-1}(h_i^3 - 1)} d(u_n, p)\\ &\leq Md(x_n, z),\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^3-1)}$. So, for all $p \in \Omega$, we get

$$d(u_n, u_m) \leq d(u_n, p) + d(u_m, p) \leq (1 + M)d(u_n, p).$$

Taking the infimum over all $p \in \Omega$, we have

$$d(u_n, u_m) \leq (1 + M)d(u_n, \Omega).$$

It follows from $\lim_{n\to\infty} d(u_n, \Omega) = 0$ that $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space X, then $\{u_n\}$ converges strongly to some $p^* \in \mathcal{K}$. It is easy to see that $F(\mathcal{S}_1)$, $F(\mathcal{S}_2)$, $F(\mathcal{S}_3)$, $F(\mathcal{T}_1)$, $F(\mathcal{T}_2)$ and $F(\mathcal{T}_3)$ are closed, that is, Ω is closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty} d(u_n, \Omega) = 0$ gives that $d(p^*, \Omega) = 0$, we have $p^* \in \Omega$. The proof is completed. □

Theorem 3.1.10 *Considering the assumption in Lemma 3.1.7 and if one of* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *and* $\mathcal{T}_3$ *is completely continuous after that the sequence* $\{u_n\}$ *defined by* (3.1.47) *converges strongly to a point in* Ω*.*

Proof. Let $\mathcal{S}_1$ be completely continuous. By Lemma 3.1.6, $\{x_n\}$ is bounded. This mean, there is a subsequence $\{\mathcal{S}_1 u_{n_j}\}$ of $\{\mathcal{S}_1 u_n\}$ such that $\{\mathcal{S}_1 u_{n_j}\}$ converges strongly to some $\xi^* \in \mathcal{K}$. Moreover, by Lemma 3.1.7, we have

$$\lim_{j\to\infty} d(u_{n_j}, \mathcal{S}_1 u_n) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{S}_2 u_n) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{S}_3 u_n) = 0 \text{ and}$$

$$\lim_{j\to\infty} d(u_{n_j}, \mathcal{T}_1 u_n) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{T}_2 u_n) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{T}_3 u_n) = 0,$$

which implies that,

$$\begin{aligned} d(u_{n_j}, \xi^*) &\leq d(u_{n_j}, \mathcal{S}_1 u_{n_j}) + d(\mathcal{S}_1 u_{n_j}, \xi^*) \\ &\to 0 \quad (\text{ as } \ j \to \infty). \end{aligned}$$

Hence $\mathcal{S}_1 u_{n_j} \to \xi^* \in \mathcal{K}$. Consequently,

$$d(\xi^*, \mathcal{S}_i \xi^*) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{S}_i u_{n_j}) = 0.$$

Since $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are continuous, for $i = 1, 2, 3$. By Lemma 3.1.7, we have

$$d(\xi^*, \mathcal{T}_i \xi^*) = \lim_{j\to\infty} d(u_{n_j}, \mathcal{T}_i u_{n_j}) = 0.$$

This implies that $\xi^* \in F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(\mathcal{S}_3) \cap F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. Using Lemma 3.1.6, we obtain $\lim_{n\to\infty} d(u_n, \xi^*)$ exists, and so $\lim_{n\to\infty} d(u_n, \xi^*) = 0$. It follows that $\{u_n\}$ converges strongly to a common fixed point of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$. The proof is completed. $\square$

CHAPTER IV

CONCLUSIONS

4.1 Conclusion

The following results are all main theorems of this thesis:

Theorem 4.1.1 *Let $\mathcal{K}$, X, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the hypotheses of Lemma 3.1.2. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and $\mathcal{S}_i$, $\mathcal{T}_i$ for all $i = 1,2,3$ satisfy the condition (ii) in Lemma 3.1.2. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f(d(x,\Omega)) \leq d(x,\mathcal{S}_1x) + d(x,\mathcal{S}_2x) + d(x,\mathcal{S}_3x) + d(x,\mathcal{T}_1x) + d(x,\mathcal{T}_2x) + d(x,\mathcal{T}_3x)$$

for all $x \in \mathcal{K}$, where $d(x,\Omega) = \inf\{d(x,v) : v \in \Omega\}$. Then the sequence $\{x_n\}$ defined by algorithm (3.1.2) converges strongly to a common fixed point of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Theorem 4.1.2 *Considering the assumption in Lemma 3.1.2 and if one of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ is completely continuous after that the sequence $\{x_n\}$ defined by 3.1.1 converges strongly to a point in Ω.*

Theorem 4.1.3 *Let $\mathcal{K}$, X, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ satisfy the hypotheses of Lemma 3.1.7, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[\varepsilon, 1-\varepsilon]$, $\exists\varepsilon \in (0,1)$, and $\mathcal{S}_i$, $\mathcal{T}_i$ for any $i = 1,2,3$ satisfy the condition (ii) in Lemma 3.1.7. Suppose that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, $\forall r \in (0,\infty)$ such that*

$$f(d(u,\Omega)) \leq d(u,\mathcal{S}_1u) + d(u,\mathcal{S}_2u) + d(u,\mathcal{S}_3u) + d(u,\mathcal{T}_1u) + d(u,\mathcal{T}_2u) + d(u,\mathcal{T}_3u),$$

$\forall u \in \mathcal{K}$, *where* $d(u,\Omega) = \inf\{d(u,p) : p \in \Omega\}$. *Then the sequence* $\{u_n\}$ *defined by* (3.1.47) *converges strongly to a common fixed point of* $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$.

Theorem 4.1.4 *Considering the assumption in Lemma 3.1.7 and if one of* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{T}_1, \mathcal{T}_2$ *and* $\mathcal{T}_3$ *is completely continuous after that the sequence* $\{u_n\}$ *defined by* (3.1.47) *converges strongly to a point in* Ω.

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Publications

1. **Paimsang, P.**, and Thianwan, T. (2023). A Novel Noor Iterative Technique for Mixed Type Asymptotically Nonexpansive Mappings in Hyperbolic Spaces. Thai Journal of Mathematics, 21(2), 413-430.

Conference presentations

1. **Paimsang, P.** (September 8 - 9, 2022). Inertial algorithm for Bregman quasi-nonexpansive mappings in real reflexive Banach spaces. International conference on Digital Image Processing and Machine Learning (ICDIPML2022), Faculty of Science, University of Phayao, Phayao, Thailand.
2. **Paimsang, P.** (june 22 - 24, 2023). A new technique of Noor iterative in hyperbolic spaces for mixed type asymptotically nonexpansive mappings. The 2023 International Conference

of the Honam-Chungcheong Mathematical Societies, Jeonbuk National University, Jeonju, South Korea.

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