INTERNAL AND EXTERNAL DIRECT PRODUCTS OF UP (BCC)-ALGEBRAS

CHATSUDA CHANMANEE

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Submitted by Chatsuda Chanmanee

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University of Phayao

....Chairman

(Acting Sub Lieutenant Dr. Pongpun Julatha)

....Committee

(Associate Professor Dr. Aiyared Iampan)

....Committee

(Associate Professor Dr. Tanakit Thianwan)

....Committee

(Associate Professor Dr. Damrongsak Yambangwai)

Approved by

....

(Associate Professor Dr. Chayan Boonyarak)

Dean of School of Science

March 2023

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เรื่อง: ผลคูณตรงภายในและภายนอกของพีชคณิตยูพี (บีซีซี)

ผู้วิจัย: ฉัตรสุดา จันทร์มณี, วิทยานิพนธ์: วท.ม. (คณิตศาสตร์), มหาวิทยาลัยพะเยา, 2566

ประธานที่ปรึกษา: รองศาสตราจารย์ ดร.อัยเรศ เอี่ยมพันธ์

กรรมการที่ปรึกษา: รองศาสตราจารย์ ดร.ธนกฤต เทียนหวาน, รองศาสตราจารย์ ดร.ดำรงศักดิ์ แย้มบางหวาย

คำสำคัญ: พีชคณิตยูพี, พีชคณิตบีซีซี, พีชคณิตดีบีซีซี, ผลคูณตรงภายนอก, ผลคูณตรงอ่อน, ผลคูณตรงภายใน, ผลคูณตรงปฏิภายใน, ฟังก์ชันสาทิสสัณฐานยูพี (บีซีซี), ฟังก์ชันปฏิสาทิสสัณฐานยูพี (บีซีซี)

บทคัดย่อ

แนวคิดของผลคูณตรงของวงศ์จำกัดของพีชคณิตบี ถูกแนะนำโดย Lingcong และ Endam ในปี ค.ศ. 2016 ในวิทยานิพนธ์นี้ เราจะแนะนำแนวคิดของผลคูณตรงของวงศ์อนันต์ของพีชคณิตยูพี (บีซีซี) เราเรียกแนวคิดนี้ว่า ผลคูณตรงภายนอก เราหาผลลัพธ์ของผลคูณตรงภายนอกของเซตย่อยพิเศษของพีชคณิตยูพี (บีซีซี) รวมทั้งเรายังแนะนำแนวคิดของผลคูณตรงอ่อนของพีชคณิตยูพี (บีซีซี) มากกว่านั้น เรายังจัดหาทฤษฎีบทพื้นฐานของฟังก์ชัน(ปฏิ)สาทิสสัณฐานยูพี (บีซีซี) ในมุมมองของผลคูณตรงภายนอกของพีชคณิตยูพี (บีซีซี) นอกเหนือจากนี้ เรายังประยุกต์แนวคิดของผลคูณตรงภายในของกรุปพอยด์ไปยังพีชคณิตยูพี (บีซีซี) โดยเราจะแนะนำสี่แนวคิดของผลคูณตรงภายในของพีชคณิตยูพี (บีซีซี) ดังนี้ ผลคูณตรงภายใน ผลคูณตรงปฏิภายใน ผลคูณตรงภายในชนิดที่ 2 และผลคูณตรงปฏิภายในชนิดที่ 2 อีกทั้งเราศึกษาคุณสมบัติของทั้งสี่แนวคิดนี้ และหาสมบัติที่จำเป็นและสำคัญสำหรับการสรุปผลในวิทยานิพนธ์นี้ สุดท้าย เราพิสูจน์ทฤษฎีบทสำคัญดังนี้ สำหรับพีชคณิตยูพี (บีซีซี) ใด ๆ จะมีเพียงรูปแบบเดียวเท่านั้นสำหรับผลคูณตรงภายใน และจะมีเพียงรูปแบบเดียวเท่านั้นสำหรับผลคูณตรงปฏิภายใน และท้ายสุดจะมีเพียงพีชคณิตยูพี (บีซีซี) ศูนย์รูปแบบเดียวเท่านั้น ที่สอดคล้องกับผลคูณตรงภายในชนิดที่ 2 และผลคูณตรงปฏิภายในชนิดที่ 2

Title: INTERNAL AND EXTERNAL DIRECT PRODUCTS OF UP (BCC)-ALGEBRAS

Author: Chatsuda Chanmanee, Thesis: M.Sc. (Mathematics), University of Phayao, 2023

Advisor: Associate Professor Dr. Aiyared Iampan

Committee: Associate Professor Dr. Tanakit Thianwan, Associate Professor Dr. Damrongsak Yambangwai

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ABSTRACT

The concept of the direct product of finite family of B-algebras is introduced by Lingcong and Endam in 2016. In this thesis, we introduce the concept of the direct product of infinite family of UP (BCC)-algebras, we call the external direct product. We find the result of the external direct product of special subsets of UP (BCC)-algebras. Also, we introduce the concept of the weak direct product UP (BCC)-algebras. Moreover, we provide the fundamental theorem of (anti-)UP (BCC)-homomorphisms in view of the external direct product UP (BCC)-algebra. In addition, we apply the concept of the internal direct product of a groupoid to a UP (BCC)-algebra in which we introduce four new concepts of internal direct products of UP (BCC)-algebras: the internal direct product, the anti-internal direct product, the internal direct product of type 2, and the anti-internal direct product of type 2. We explore the properties of four concepts and find the necessary and important properties for concluding the study. Finally, we prove the important theorem that for a UP (BCC)-algebra, there can only be one form of the internal direct product and only one form of the anti-internal direct product, and finally there can only be the zero UP (BCC)-algebra that satisfies of internal direct product and the anti-internal direct product of type 2.

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CHAPTER I

INTRODUCTION

Algebraic structures are important in mathematics, whose applications have been widely used in many fields. For example, theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological space, etc. Among many algebraic structures, algebras of logic form an important class of algebras. Examples of these are BCK-algebras [16], BCI-algebras [17], UP-algebras [11], fully UP-semigroups [12], topological UP-algebras [35], UP-hyper-algebras [14], extension of KU/UP-algebras [34] and others. They are strongly connected with logic. For example, BCI-algebras were introduced by Iséki [17] in 1966 and have connections with BCI-logic, being the BCI-system in combinatory logic, which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [16, 17] in 1966 and have been extensively investigated by many researchers.

A UP-algebra is a new type of algebraic logic that has been published in international academic journals. In 2017, Iampan [11] introduced the concept of a UP-algebra. Thai researchers were interested in studying UP-algebra structures to define new structures. For example, fully UP-semigroup [12], topological UP-algebra [17], UP-hyperalgebra [14], and extension of KU/UP-algebra [34]. In 2022, Jun et al. [19] have shown that the concepts of UP-algebras and BCC-algebras are the same concept (see [11, 27]). Therefore, in this thesis and future research, our research team will use the name BCC instead of UP in honor of Komori, who first defined it in 1984.

The concept of the direct product [41] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group, and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [29, 30] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms, respectively. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra. In the same year, Endam and Teves [8] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [45] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [41] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [25] defined the direct product of GK algebra. They derived the finite form of the direct product of GK algebra. They investigated and applied the concept of the direct product of GK algebra in GK kernel. In 2022, Chanmanee et al. introduced the concept of the direct product of an infinite family of B-algebras [4] and IUP-algebras [5], they called them the external direct product. Furthermore, in 2023, Chanmanee et al. [6] introduced the concept of the external direct products of BP- algebras.

The internal direct products [44] is type of "direct product", that is to say a group is isomorphic to the direct product of two of its subgroups. Its is continually applied to other algebraic structures. In 1992, Makamba [31] shown that the internal direct product of two fuzzy subgroups is isomorphic to their external direct product. In 1999, Pledger [33] generalized the internal direct product from groups to all groupoids (binary systems). Then, it develops what seems to be a natural basic definition of the internal direct product. In 2000, Jakubík and Csontóová [18] introduced two-factor internal direct product decompositions of a connected partially ordered set. In 2012, Kamuti [23] introduced the cycle index of semidirect products, namely Frobenius groups, and discussed a very special case of semidirect products called internal direct products. In 2015, Karaçal and Khadjiev [24] introduced some relations between an external direct product and an internal direct product of a family of integral $\vee$-distributive binary aggregation functions. In 2017, Lingcong [28] introduced the internal direct product of normal subalgebras. In the same year, Nama [32] introduced the concept of a fuzzy internal direct product of fuzzy subgroups of group. In 2019, Shalla and Olgun [42] introduced neutrosophic extended triplet internal direct product and neutrosophic extended triplet external direct products of NET group. Then, they defined NET internal and external semidirect products for NET group by utilizing the notion of NET set theory of Smarandache.

From the concept of the direct product and the internal direct product mentioned above. The researcher is interested and motivated to study the direct product of an infinite family of BCC-algebras, called the external direct product. Then, we apply the concept of the internal direct product of an algebra to a BCC-algebra, called the internal direct products of BCC-algebras. The content is divided into 4 chapters: Introduction, Preliminaries, Main results, and Conclusions. Chapter 2 introduces the definitions, properties, and examples required in this thesis. In Chapter 3, we divided into 5 sections as follows: external direct products of BCC-algebras, external direct products of dual BCC-algebras, internal direct products of BCC-algebras, anti-internal direct products of BCC-algebras, and internal direct products of type two of BCC-algebras. In the final thesis chapter, we summarize and reflect on the main findings of this thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we introduce the definitions, properties, and examples of BCC-algebras and dBCC-algebras required in this thesis as follows:

The concept of BCC-algebras (see [27]) can be redefined without the condition (2.0.1) as follows:

Definition 2.0.1 [10] An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *BCC-algebra* if it satisfies the following axioms:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0), \tag{BCC-1}$$

$$(\forall x \in X)(0 * x = x), \tag{BCC-2}$$

$$(\forall x \in X)(x * 0 = 0), \tag{BCC-3}$$

$$(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y). \tag{BCC-4}$$

If the BCC-algebra has only one element 0, then we call it the *zero BCC-algebra*.

Example 2.0.2 Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	4	0	6
2	0	1	0	3	4	5	6
3	0	1	2	0	4	0	6
4	0	1	2	3	0	5	6
5	0	1	2	3	4	0	6
6	0	1	2	3	4	5	0

Then $X = (X; *, 0)$ is a BCC-algebra.

For more examples of BCC-algebras, see [2, 3, 7, 12, 15, 26, 36, 37, 39, 40].

Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be BCC-algebras. A map $\varphi : A \to B$ is called a *BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker\varphi$, is defined to be the $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker\varphi$ is a BCC-ideal of A, and $\ker\varphi = \{0_A\}$ if and only if φ is injective. A (anti-)BCC-homomorphism φ is called a (anti-)BCC-monomorphism, (anti-) BCC-epimorphism, or (anti-)BCC-isomorphism if φ is injective, surjective, or bijective, respectively.

In a BCC-algebra $X = (X; *, 0)$, the following assertions are valid (see [11, 12]).

$$(\forall x \in X)(x * x = 0), \tag{2.0.1}$$

$$(\forall x, y, z \in X)(x * y = 0, y * z = 0 \Rightarrow x * z = 0),$$

$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (z * x) * (z * y) = 0),$$

$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (y * z) * (x * z) = 0),$$

$$(\forall x, y \in X)(x * (y * x) = 0),$$

$$(\forall x, y \in X)((y * x) * x = 0 \Leftrightarrow x = y * x),$$

$$(\forall x, y \in X)(x * (y * y) = 0),$$

$$(\forall u, x, y, z \in X)((x * (y * z)) * (x * ((u * y) * (u * z))) = 0),$$

$$(\forall u, x, y, z \in X)((((u * x) * (u * y)) * z) * ((x * y) * z) = 0),$$

$$(\forall x, y, z \in X)(((x * y) * z) * (y * z) = 0),$$

$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow x * (z * y) = 0),$$

$$(\forall x, y, z \in X)(((x * y) * z) * (x * (y * z)) = 0),$$

$$(\forall u, x, y, z \in X)(((x * y) * z) * (y * (u * z)) = 0).$$

According to [11], the binary relation $\leq$ on a BCC-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Definition 2.0.3 A BCC-algebra $X = (X; *, 0)$ is said to be

(i) *bounded* if there is an element $1 \in X$ such that $1 \leq x$ for all $x \in X$, that is,

$$(\forall x \in X)(1 * x = 0), \qquad \text{(Bounded)}$$

(ii) *meet-commutative* [38] if it satisfies the identity

$$(\forall x, y \in X)(x \wedge y = y \wedge x), \qquad \text{(Meet-commutative)}$$

where

$$(\forall x, y \in X)(x \wedge y = (y * x) * x). \qquad \text{(Meet)}$$

Example 2.0.4 Let $X = \{0, 1, 2, 3\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	0	0	0

Then $X = (X; *, 0)$ is a bounded BCC-algebra.

Example 2.0.5 Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	1	2	0	4
4	0	1	2	3	0

Then $X = (X; *, 0)$ is a meet-commutative BCC-algebra.

Definition 2.0.6 A nonempty subset S of a BCC-algebra $X = (X; *, 0)$ is called

(i) a *BCC-subalgebra* [11] of X if it satisfies the following condition:

$$(\forall x, y \in S)(x * y \in S), \tag{2.0.2}$$

(ii) a *near BCC-filter* [13] of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x * y \in S), \tag{2.0.3}$$

(*iii*) a *BCC-filter* [43] of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{2.0.4}$$

$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S), \tag{2.0.5}$$

(*iv*) an *implicative BCC-filter* [21] of X if it satisfies the condition (2.0.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S), \tag{2.0.6}$$

(*v*) a *comparative BCC-filter* [20] of X if it satisfies the condition (2.0.4) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S), \tag{2.0.7}$$

(*vi*) a *shift BCC-filter* [22] of X if it satisfies the condition (2.0.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S), \tag{2.0.8}$$

(*vii*) a *BCC-ideal* [11] of X if it satisfies the condition (2.0.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{2.0.9}$$

(*viii*) a *strong BCC-ideal* [9] of X if it satisfies the condition (2.0.4) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{2.0.10}$$

We know that the concept of BCC-subalgebras is a generalization of near BCC-filters, near BCC-filters is a generalization of BCC-filters, BCC-filters is a generalization of BCC-ideals, BCC-filters is a generalization of implicative BCC-filters, BCC-filters is a generalization of comparative BCC-filters, BCC-filters is a generalization of shift BCC-filters, BCC-ideals is a generalization of implicative BCC-filters, implicative BCC-filters is a generalization of strong BCC-ideals, comparative BCC-filters is a generalization of strong BCC-ideals, shift BCC-filters is a generalization of strong BCC-ideals. Moreover, a BCC-algebra X is the only strong BCC-ideal. We get the diagram of the special subsets of BCC-algebras, which is shown in Figure 1.

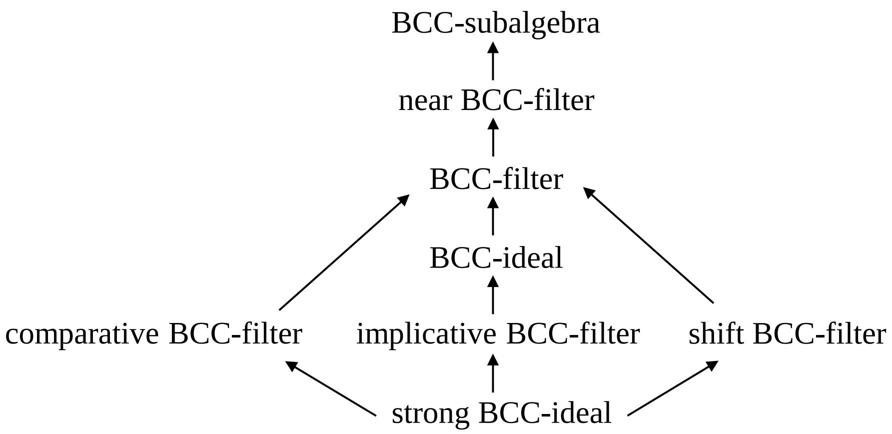

Figure 1: Special subsets of BCC-algebras

Definition 2.0.7 An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *dual BCC-algebra* (dBCC-algebra) if it satisfies (BCC-4) and the following axioms:

$$(\forall x, y, z \in X)((((z * x) * (y * x)) * (z * y) = 0), \qquad \text{(dBCC-1)}$$

$$(\forall x \in X)(x * 0 = x), \qquad \text{(dBCC-2)}$$

$$(\forall x \in X)(0 * x = 0). \qquad \text{(dBCC-3)}$$

The binary relation $\leq$ on a dBCC-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Example 2.0.8 Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	1
2	2	0	0	3	3	3	2
3	3	0	0	0	3	3	3
4	4	0	0	0	0	4	4
5	5	0	0	0	0	0	5
6	6	0	0	0	0	0	0

Then $X = (X; *, 0)$ is a dBCC-algebra.

Definition 2.0.9 A dBCC-algebra $X = (X; *, 0)$ is said to be

(i) *bounded* if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$, that is,

$$(\forall x \in X)(x * 1 = 0), \qquad \text{(Bounded)}$$

(ii) *join-commutative* if it satisfies the identity

$$(\forall x, y \in X)(x \vee y = y \vee x), \qquad \text{(Join-commutative)}$$

where

$$(\forall x, y \in X)(x \vee y = x * (x * y)). \qquad \text{(Join)}$$

Example 2.0.10 Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	0	0	3	2
3	3	0	0	0	3
4	4	0	0	0	0

Then $X = (X; *, 0)$ is a bounded dBCC-algebra.

Example 2.0.11 Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	0	0	2	2
3	3	0	3	0	3
4	4	4	4	4	0

Then $X = (X; *, 0)$ is a join-commutative dBCC-algebra.

Definition 2.0.12 A nonempty subset S of a dBCC-algebra $X = (X; *, 0)$ is called

(i) a *dBCC-subalgebra* of X if it satisfies the condition (2.0.2).

(ii) a *near dBCC-filter* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow y * x \in S), \tag{2.0.11}$$

(iii) a *dBCC-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{2.0.12}$$

$$(\forall x, y \in X)(y * x \in S, x \in S \Rightarrow y \in S), \tag{2.0.13}$$

(iv) an *implicative dBCC-filter* of X if it satisfies the condition (2.0.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S), \tag{2.0.14}$$

(v) a *comparative dBCC-filter* of X if it satisfies the condition (2.0.12) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S), \tag{2.0.15}$$

(vi) a *shift dBCC-filter* of X if it satisfies the condition (2.0.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S), \tag{2.0.16}$$

(vii) a *dBCC-ideal* of X if it satisfies the condition (2.0.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{2.0.17}$$

$(viii)$ a *strong dBCC-ideal* of X if it satisfies the condition (2.0.12) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{2.0.18}$$

CHAPTER III

MAIN RESULTS

3.1 External direct products of BCC-algebras

Lingcong and Endam [29] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebras to a finite family of B-algebras, and some of the related properties were investigated as follows:

Definition 3.1.1 [29] Let $(X_i; *_i)$ be an algebra for each $i \in \{1, 2, ..., k\}$. Define the *direct product* of algebras $X_1, X_2, ..., X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times ... \times X_k = \{(x_1, x_2, ..., x_k) \mid x_i \in X_i,\ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, ..., x_k) \otimes (y_1, y_2, ..., y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, ..., x_k *_k y_k)$$

for all $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in \prod_{i=1}^{k} X_i$.

Now, we extend the concept of the direct product to an infinite family of BCC-algebras and provide some of its properties.

Definition 3.1.2 Let X_i be a nonempty set for each $i \in I$. Define the *external direct product* of sets X_i for all $i \in I$ to be the set $\prod_{i \in I} X_i$, where

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i,\ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i\in I} X_i$ with a function $(x_i)_{i\in I} : I \to \bigcup_{i\in I} X_i$, where $i \mapsto x_i \in X_i$ for all $i \in I$.

Remark 3.1.3 Let X_i be a nonempty set and S_i a subset of X_i for all $i \in I$. Then $\prod_{i\in I} S_i$ is a nonempty subset of the external direct product $\prod_{i\in I} X_i$ if and only if S_i is a nonempty subset of X_i for all $i \in I$.

Definition 3.1.4 Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\otimes$ on the external direct product $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes)$ as follows:

$$(\forall (x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i)((x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I}).$$

We shall show that $\otimes$ is a binary operation on $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $*_i$ is a binary operation on X_i, we have $x_i *_i y_i \in X_i$ for all $i \in I$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x_i')_{i\in I}, (y_i')_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x_i')_{i\in I} = (y_i')_{i\in I}$. We shall show that $(x_i)_{i\in I} \otimes (x_i')_{i\in I} = (y_i)_{i\in I} \otimes (y_i')_{i\in I}$. Then

$$x_i = y_i \text{ and } x_i' = y_i' \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i, we have $x_i *_i x_i' = y_i *_i y_i'$ for all $i \in I$. Thus

$$\begin{aligned}(x_i)_{i\in I} \otimes (x_i')_{i\in I} &= (x_i *_i x_i')_{i\in I} \\ &= (y_i *_i y_i')_{i\in I} \\ &= (y_i)_{i\in I} \otimes (y_i')_{i\in I}.\end{aligned}$$

Hence, $\otimes$ is a binary operation on $\prod_{i\in I} X_i$.

Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(f_{x_i}(j) = \begin{cases} x_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right). \tag{3.1.1}$$

Then $f_{x_i} \in \prod_{i\in I} X_i$.

Remark 3.1.5 Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, we have $f_{0_i} = (0_i)_{i\in I}$.

Lemma 3.1.6 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (2.0.1), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

By (3.1.1), we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$. □

The following theorem shows that the direct product of BCC-algebras in terms of an infinite family of BCC-algebras is also a BCC-algebra.

Theorem 3.1.7 *$X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$.

(BCC-1) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-1), we have $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$ for all $i \in I$. Thus

$$\begin{aligned}
&((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I}))\\
&= (y_i *_i z_i)_{i\in I} \otimes ((x_i *_i y_i)_{i\in I} \otimes (x_i *_i z_i)_{i\in I})\\
&= (y_i *_i z_i)_{i\in I} \otimes ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I}\\
&= ((y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)))_{i\in I}\\
&= (0_i)_{i\in I}.
\end{aligned}$$

(BCC-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(0_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(BCC-3) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-3), we have $x_i *_i 0_i = 0_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \otimes (0_i)_{i\in I} = (x_i *_i 0_i)_{i\in I} = (0_i)_{i\in I}.$$

(BCC-4) Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i)_{i\in I}$. Then $(x_i *_i y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$, so $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Since X_i satisfies (BCC-4), we have $x_i = y_i$ for all $i \in I$. Therefore, $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4. Let $i \in I$.

(BCC-1) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-1), we have $(f_{y_i} \otimes f_{z_i}) \otimes ((f_{x_i} \otimes f_{y_i}) \otimes (f_{x_i} \otimes f_{z_i})) = (0_i)_{i\in I}$. By Remark 3.1.5 and Lemma 3.1.6, we get $f_{(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i))} = f_{0_i}$. It follows from (3.1.1) that $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$.

(BCC-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-2), we have $(0_i)_{i\in I} \otimes f_{x_i} = f_{x_i}$. By Remark 3.1.5 and Lemma 3.1.6, we get $f_{0_i *_i x_i} = f_{x_i}$. It follows from (3.1.1) that $0_i *_i x_i = x_i$.

(BCC-3) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-3), we have $f_{x_i} \otimes (0_i)_{i\in I} = (0_i)_{i\in I}$. By Remark 3.1.5 and Lemma 3.1.6, we get $f_{x_i *_i 0_i} = f_{0_i}$. It follows from (3.1.1) that $x_i *_i 0_i = 0_i$.

(BCC-4) Let $x_i, y_i \in X_i$ be such that $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By assumption and (2.0.1), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

Thus $f_{x_i} \otimes f_{y_i} = (0_i)_{i\in I}$ and $f_{y_i} \otimes f_{x_i} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ satisfies (BCC-4), we have $f_{x_i} = f_{y_i}$. Therefore, $x_i = y_i$. Hence, $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$. □

We call the BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ in Theorem 3.1.7 the external direct product BCC-algebra induced by a BCC-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Theorem 3.1.8 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a bounded BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4.*

Proof. By Theorem 3.1.7, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4. We are left to prove that X_i is bounded for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is bounded.

Assume that X_i is bounded for all $i \in I$. Then for all $i \in I$, there exists $1_i \in X_i$ such that $1_i \leq x_i$ for all $x_i \in X_i$. That is, $1_i * x_i = 0_i$ for all $i \in I$. Now, $(1_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Thus

$$(1_i)_{i\in I} \otimes (x_i)_{i\in I} = (1_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

That is, $(1_i)_{i\in I} \leq (x_i)_{i\in I}$. Hence, $\prod_{i\in I} X_i$ is bounded.

Conversely, assume that $\prod_{i\in I} X_i$ is bounded. Then for all $i \in I$, there exists $(1_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $(1_i)_{i\in I} \leq (x_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. That is, $(1_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $i \in I$. Now, $1_i \in X_i$. Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ is bounded, we have $(1_i)_{i\in I} \otimes f_{x_i} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I) \left(((1_i)_{i\in I} \otimes f_{x_i})(j) = \begin{cases} 1_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases} \right),$$

this implies that $1_i *_i x_i = 0_i$. That is, $1_i \leq x_i$. Hence, X_i is bounded for all $i \in I$. □

Theorem 3.1.9 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a meet-commutative BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a meet-commutative BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4.*

Proof. By Theorem 3.1.7, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 3.1.4. We are left to prove that X_i is meet-commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is meet-commutative.

Assume that X_i is meet-commutative for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is meet-commutative, we have $x_i \wedge y_i = y_i \wedge x_i$ for all $i \in I$. That is, $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$ for all $i \in I$. Thus

$$\begin{aligned}
(x_i)_{i\in I} \wedge (y_i)_{i\in I} &= ((y_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes (x_i)_{i\in I} \\
&= (y_i *_i x_i)_{i\in I} \otimes (x_i)_{i\in I} \\
&= ((y_i *_i x_i) *_i x_i)_{i\in I} \\
&= ((x_i *_i y_i) *_i y_i)_{i\in I} \\
&= (x_i *_i y_i)_{i\in I} \otimes (y_i)_{i\in I} \\
&= ((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I}
\end{aligned}$$

$$= (y_i)_{i\in I} \wedge (x_i)_{i\in I}.$$

Hence, $\prod_{i\in I} X_i$ is meet-commutative.

Conversely, assume that $\prod_{i\in I} X_i$ is meet-commutative. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Since $\prod_{i\in I} X_i$ is meet-commutative, we have $f_{x_i} \wedge f_{y_i} = f_{y_i} \wedge f_{x_i}$. That is, $(f_{y_i} \otimes f_{x_i}) \otimes f_{x_i} = (f_{x_i} \otimes f_{y_i}) \otimes f_{y_i}$. By Lemma 3.1.6, we have $f_{(y_i *_i x_i) *_i x_i} = f_{(x_i *_i y_i) *_i y_i}$. By (3.1.1), we have $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$. Hence, X_i is meet-commutative for all $i \in I$. $\square$

Next, we introduce the concept of the weak direct product of an infinite family of BCC-algebras and obtain some of its properties as follows:

Definition 3.1.10 Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Define the *weak direct product* of a BCC-algebra X_i for all $i \in I$ to be the structure $\prod^{\mathrm{w}}_{i\in I} X_i = (\prod^{\mathrm{w}}_{i\in I} X_i; \otimes)$, where

$$\prod^{\mathrm{w}}_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 3.1.11 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is a BCC-subalgebra of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite.

Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j y_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (BCC-2) and (BCC-3), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ y_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 3.1.12 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-subalgebra of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (2.0.2), we have $x_i *_i y_i \in S_i$ for all $i \in I$, so $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$. Since

$\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By (2.0.2) and Lemma 3.1.6, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a BCC-subalgebra of X_i for all $i \in I$. □

Theorem 3.1.13 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a near BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a near BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a near BCC-filter of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i \in S_i$ for all $i \in I$, it follows from (2.0.3) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a near BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a near BCC-filter of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By (2.0.3) and Lemma 3.1.6, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a near BCC-filter of X_i for all $i \in I$. □

Theorem 3.1.14 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that

$(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (2.0.5) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (2.0.5), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $y_i \in S_i$. Hence, S_i is a BCC-filter of X_i for all $i \in I$. □

Theorem 3.1.15 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an implicative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an implicative BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an implicative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (2.0.6) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an implicative BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an implicative BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (2.0.6) and Lemma 3.1.6, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in$

$\prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is an implicative BCC-filter of X_i for all $i \in I$. □

Theorem 3.1.16 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a comparative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a comparative BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a comparative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (y_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i ((y_i *_i z_i) *_i y_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$, it follows from (2.0.7) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a comparative BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a comparative BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $f_{x_i} \otimes ((f_{y_i} \otimes f_{z_i}) \otimes f_{y_i}) = f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$. By (2.0.7), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $y_i \in S_i$. Hence, S_i is a comparative BCC-filter of X_i for all $i \in I$. □

Theorem 3.1.17 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a shift BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a shift BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a shift BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then

$(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$, it follows from (2.0.8) that $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$ for all $i \in I$. Thus $(((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I}) \otimes (z_i)_{i\in I} = (((z_i *_i y_i) *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a shift BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a shift BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (2.0.8) and Lemma 3.1.6, we have $f_{((z_i *_i y_i) *_i y_i) *_i z_i} = ((f_{z_i} \otimes f_{y_i}) \otimes f_{y_i}) \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$. Hence, S_i is a shift BCC-filter of X_i for all $i \in I$. □

Theorem 3.1.18 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-ideal of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$, it follows from (2.0.9) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (2.0.9) and Lemma

3.1.6, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is a BCC-ideal of X_i for all $i \in I$. $\square$

Theorem 3.1.19 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong BCC-ideal of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a strong BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((z_i *_i y_i) *_i (z_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$, it follows from (2.0.10) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong BCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong BCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.1.6, we have $(f_{z_i} \otimes f_{y_i}) \otimes (f_{z_i} \otimes f_{x_i}) = f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$. By (2.0.10), we have $f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i \in S_i$. Hence, S_i is a strong BCC-ideal of X_i for all $i \in I$. $\square$

Moreover, we discuss several BCC-homomorphism theorems in view of the external direct product of BCC-algebras.

Definition 3.1.20 Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}). \tag{3.1.2}$$

We shall show that $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $\psi_i : X_i \to S_i$ is a function and $x_i \in X_i$, we have $\psi_i(x_i) \in S_i$ for all $i \in I$. Thus $(\psi_i(x_i))_{i\in I} \in \prod_{i\in I} S_i$ such that $\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}$. Let $(x_i)_{i\in I}, (x_i')_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (x_i')_{i\in I}$. Then $x_i = x_i'$, so $\psi_i(x_i) = \psi_i(x_i')$ for all $i \in I$. Thus

$$\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I} = (\psi_i(x_i'))_{i\in I} = \psi(x_i')_{i\in I}.$$

Therefore, $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function.

Theorem 3.1.21 *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 3.1.20,

(ii) ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,

(iii) ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.

Proof. (i) Assume that ψ_i is injective for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $\psi(x_i)_{i\in I} = \psi(y_i)_{i\in I}$. Then $(\psi_i(x_i))_{i\in I} = (\psi_i(y_i))_{i\in I}$. Thus $\psi_i(x_i) = \psi_i(y_i)$ for all $i \in I$. Since ψ_i is injective, we have $x_i = y_i$ for all $i \in I$. Thus $(x_i)_{i\in I} = (y_i)_{i\in I}$. Hence, ψ is injective.

Conversely, assume that ψ is injective. Let $i \in I$. Let $x_i, x_i' \in X_i$ be such that $\psi_i(x_i) = \psi_i(x_i')$.

Let $x_j = x_j' \in X_j$ for all $j \in I$ and $j \neq i$. Then $\psi_j(x_j) = \psi_j(x_j') \in S_j$.

Let $h_{\psi_i(x_i)} : I \to \bigcup_{i\in I} S_i$ and $h_{\psi_i(x'_i)} : I \to \bigcup_{i\in I} S_i$ are functions defined by

$$(\forall j \in I) \left(h_{\psi_i(x_i)}(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(x_j) & \text{otherwise} \end{cases} \right) \tag{3.1.3}$$

and

$$(\forall j \in I) \left(h_{\psi_i(x'_i)}(j) = \begin{cases} \psi_i(x'_i) & \text{if } j = i \\ \psi_j(x'_j) & \text{otherwise} \end{cases} \right). \tag{3.1.4}$$

Then $h_{\psi_i(x_i)}, h_{\psi_i(x'_i)} \in \prod_{i\in I} S_i$ such that $\psi(x_i)_{i\in I} = h_{\psi_i(x_i)} = h_{\psi_i(x'_i)} = \psi(x'_i)_{i\in I}$. Since ψ is injective, we have $(x_i)_{i\in I} = (x'_i)_{i\in I}$. Thus $x_i = x'_i$. Hence, ψ_i is injective for all $i \in I$.

(ii) Assume that ψ_i is surjective for all $i \in I$. Let $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $y_i \in S_i$ for all $i \in I$. Since ψ_i is surjective, there exists $x_i \in X_i$ such that $\psi_i(x_i) = y_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I} = (y_i)_{i\in I}.$$

Hence, ψ is surjective.

Conversely, assume that ψ is surjective. Let $i \in I$. Let $k_i \in S_i$.

Let $k_j \in S_j$ for all $j \in I$ and $j \neq i$. Then $(k_i)_{i\in I} \in \prod_{i\in I} S_i$. Since ψ is surjective, there exists $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(k_i)_{i\in I} = \psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}.$$

Thus $k_i = \psi_i(x_i)$. Hence, ψ_i is surjective for all $i \in I$.

(iii) It is straightforward from *(i)* and *(ii)*. □

Theorem 3.1.22 *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and*

$\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then

(*i*) *ψ_i is a BCC-homomorphism for all $i \in I$ if and only if ψ is a BCC-homomorphism which is defined in Definition 3.1.20,*

(*ii*) *ψ_i is a BCC-monomorphism for all $i \in I$ if and only if ψ is a BCC-monomorphism,*

(*iii*) *ψ_i is a BCC-epimorphism for all $i \in I$ if and only if ψ is a BCC-epimorphism,*

(*iv*) *ψ_i is a BCC-isomorphism for all $i \in I$ if and only if ψ is a BCC-isomorphism,*

(*v*) $\ker \psi = \prod_{i \in I} \ker \psi_i$ *and* $\psi(\prod_{i \in I} X_i) = \prod_{i \in I} \psi_i(X_i)$.

Proof. (*i*) Assume that ψ_i is a BCC-homomorphism for all $i \in I$. Let $(x_i)_{i \in I}$, $(x'_i)_{i \in I} \in \prod_{i \in I} X_i$. Then

$$\begin{aligned}
\psi((x_i)_{i \in I} \otimes (x'_i)_{i \in I}) &= \psi(x_i *_i x'_i)_{i \in I} \\
&= (\psi_i(x_i *_i x'_i))_{i \in I} \\
&= (\psi_i(x_i) \circ_i \psi_i(x'_i))_{i \in I} \\
&= (\psi_i(x_i))_{i \in I} \otimes (\psi_i(x'_i))_{i \in I} \\
&= \psi(x_i)_{i \in I} \otimes \psi(x'_i)_{i \in I}.
\end{aligned}$$

Hence, ψ is a BCC-homomorphism.

Conversely, assume that ψ is a BCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i \in I} X_i$, which is defined by (3.1.1). Since ψ is a

BCC-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{x_i}) \otimes \psi(f_{y_i})$. Since

$$(\forall j \in I) \left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases} \right),$$

we have

$$(\forall j \in I) \left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases} \right). \tag{3.1.5}$$

Since

$$(\forall j \in I) \left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right)$$

and

$$(\forall j \in I) \left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases} \right),$$

we have

$$(\forall j \in I) \left((\psi(f_{x_i}) \otimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases} \right). \tag{3.1.6}$$

By (3.1.5) and (3.1.6), we have $\psi_i(x_i *_i y_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is a BCC-homomorphism for all $i \in I$.

(*ii*) It is straightforward from (*i*) and Theorem 3.1.21 (*i*).

(*iii*) It is straightforward from (*i*) and Theorem 3.1.21 (*ii*).

(iv) It is straightforward from (i) and Theorem 3.1.21 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I\\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I\\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I}\\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I}\\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I\\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

In what follows, we discuss several anti-BCC-homomorphism theorems in view of the external direct product of BCC-algebras.

Theorem 3.1.23 *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) ψ_i is an anti-BCC-homomorphism for all $i \in I$ if and only if ψ is an anti-BCC-homomorphism which is defined in Definition 3.1.20,

(ii) ψ_i is an anti-BCC-monomorphism for all $i \in I$ if and only if ψ is an anti-BCC-monomorphism,

(iii) ψ_i *is an anti-BCC-epimorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-epimorphism,*

(iv) ψ_i *is an anti-BCC-isomorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-isomorphism.*

Proof. (i) Assume that ψ_i is an anti-BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I} \\ &= (\psi_i(x_i *_i x'_i))_{i\in I} \\ &= (\psi_i(x'_i) \circ_i \psi_i(x_i))_{i\in I} \\ &= (\psi_i(x'_i))_{i\in I} \otimes (\psi_i(x_i))_{i\in I} \\ &= \psi(x'_i)_{i\in I} \otimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-BCC-homomorphism.

Conversely, assume that ψ is an anti-BCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Since ψ is an anti-BCC-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{y_i}) \otimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise}\end{cases}\right). \tag{3.1.7}$$

Since

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{y_i}) \otimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{3.1.8}$$

By (3.1.7) and (3.1.8), we have $\psi_i(x_i *_i y_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an anti-BCC-homomorphism for all $i \in I$.

(*ii*) It is straightforward from (*i*) and Theorem 3.1.21 (*i*).

(*iii*) It is straightforward from (*i*) and Theorem 3.1.21 (*ii*).

(*iv*) It is straightforward from (*i*) and Theorem 3.1.21 (*iii*). □

3.2 External direct products of dual BCC-algebras

In this section, we will define a new binary operator for the external direct product. That is, we will use binary operation ⊠, which is defined in the following definitions:

Definition 3.2.1 Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\boxtimes$ on the external direct product $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes)$ as follows:

$$(\forall (x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i)((x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (y_i *_i x_i)_{i\in I}).$$

We shall show that $\boxtimes$ is a binary operation on $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}$, $(y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $*_i$ is a binary operation on X_i, we have $y_i *_i x_i \in X_i$ for all $i \in I$. Then $(y_i *_i x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (y_i *_i x_i)_{i\in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x_i')_{i\in I}, (y_i')_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x_i')_{i\in I} = (y_i')_{i\in I}$. We shall show that $(x_i)_{i\in I} \boxtimes (x_i')_{i\in I} = (y_i)_{i\in I} \boxtimes (y_i')_{i\in I}$. Then

$$x_i = y_i \text{ and } x_i' = y_i' \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i, we have $x_i' *_i x_i = y_i' *_i y_i$ for all $i \in I$. Thus

$$\begin{aligned}(x_i)_{i\in I} \boxtimes (x_i')_{i\in I} &= (x_i' *_i x_i)_{i\in I}\\ &= (y_i' *_i y_i)_{i\in I}\\ &= (y_i)_{i\in I} \boxtimes (y_i')_{i\in I}.\end{aligned}$$

Hence, $\boxtimes$ is a binary operation on $\prod_{i\in I} X_i$.

Lemma 3.2.2 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \boxtimes f_{y_i} = f_{y_i *_i x_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right).$$

By (2.0.1), we have

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

By (3.1.1), we have $f_{x_i} \boxtimes f_{y_i} = f_{y_i *_i x_i}$. □

The following theorem shows that the direct product of BCC-algebras in terms of an infinite family of BCC-algebras is also.

Theorem 3.2.3 *$X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$.

(dBCC-1) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-1), we have $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$ for all $i \in I$. Thus

$$\begin{aligned}
&(((z_i)_{i\in I} \boxtimes (x_i)_{i\in I}) \boxtimes ((y_i)_{i\in I} \boxtimes (x_i)_{i\in I}))) \boxtimes ((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \\
&= ((x_i *_i z_i)_{i\in I} \boxtimes (x_i *_i y_i)_{i\in I}) \boxtimes (y_i *_i z_i)_{i\in I} \\
&= ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I} \boxtimes (y_i *_i z_i)_{i\in I} \\
&= ((y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)))_{i\in I} \\
&= (0_i)_{i\in I}.
\end{aligned}$$

(dBCC-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \boxtimes (0_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(dBCC-3) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-3), we have $x_i *_i 0_i = 0_i$ for all $i \in I$. Thus

$$(0_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i 0_i)_{i\in I} = (0_i)_{i\in I}.$$

(BCC-4) Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (0_i)_{i\in I}$. Then $(y_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$ and $(x_i *_i y_i)_{i\in I} = (0_i)_{i\in I}$, so $y_i *_i x_i = 0_i$ and $x_i *_i y_i = 0_i$ for all $i \in I$. Since X_i satisfies (BCC-4), we have $x_i = y_i$ for all $i \in I$. Therefore, $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1. Let $i \in I$.

(BCC-1) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (dBCC-1), we have $((f_{z_i} \boxtimes f_{x_i}) \boxtimes (f_{y_i} \boxtimes f_{x_i})) \boxtimes (f_{z_i} \boxtimes f_{y_i}) = (0_i)_{i\in I}$. By Remark 3.1.5 and Lemma 3.2.2, we get $f_{(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i))} = f_{0_i}$. It follows from (3.1.1) that $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$.

(BCC-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (dBCC-2), we have $f_{x_i} \boxtimes (0_i)_{i\in I} = f_{x_i}$. By Remark 3.1.5 and Lemma 3.2.2, we get $f_{0_i *_i x_i} = f_{x_i}$. It follows from (3.1.1) that $0_i *_i x_i = x_i$.

(BCC-3) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ satisfies (dBCC-3), we have $(0_i)_{i\in I} \boxtimes f_{x_i} = (0_i)_{i\in I}$. By Remark 3.1.5 and Lemma 3.2.2, we get $f_{x_i *_i 0_i} = f_{0_i}$. It follows from (3.1.1) that $x_i *_i 0_i = 0_i$.

(BCC-4) Let $x_i, y_i \in X_i$ be such that $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$. Then

$f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \boxtimes f_{x_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By assumption and (2.0.1), we have

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \boxtimes f_{x_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

Thus $f_{x_i} \boxtimes f_{y_i} = (0_i)_{i\in I}$ and $f_{y_i} \boxtimes f_{x_i} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ satisfies (BCC-4), we have $f_{x_i} = f_{y_i}$. Therefore, $x_i = y_i$.

Hence, $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$. □

We call the dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ in Theorem 3.2.3 the external direct product dBCC-algebra induced by a BCC-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Theorem 3.2.4 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a bounded dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1.*

Proof. By Theorem 3.2.3, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1. We are left to prove that X_i is bounded for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is bounded.

Assume that X_i is bounded for all $i \in I$. Then for all $i \in I$, there exists $1_i \in X_i$ such that $1_i \leq x_i$ for all $x_i \in X_i$. That is, $1_i *_i x_i = 0_i$ for all $i \in I$. Now, $(1_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Thus

$$(x_i)_{i\in I} \boxtimes (1_i)_{i\in I} = (1_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

That is, $(x_i)_{i\in I} \leq (1_i)_{i\in I}$. Hence, $\prod_{i\in I} X_i$ is bounded.

Conversely, assume that $\prod_{i\in I} X_i$ is bounded. Then there exists $(1_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $(x_i)_{i\in I} \leq (1_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. That is, $(x_i)_{i\in I} \boxtimes (1_i)_{i\in I} = (0_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $i \in I$. Now, $1_i \in X_i$. Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since $\prod_{i\in I} X_i$ is bounded, we have $f_{x_i} \boxtimes (1_i)_{i\in I} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes (1_i)_{i\in I})(j) = \begin{cases} 1_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $1_i *_i x_i = 0_i$. That is, $1_i \leq x_i$. Hence, X_i is bounded for all $i \in I$. □

Theorem 3.2.5 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a meet-commutative BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a join-commutative dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1.*

Proof. By Theorem 3.2.3, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$

if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 3.2.1. We are left to prove that X_i is meet-commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is join-commutative.

Assume that X_i is meet-commutative for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is meet-commutative, we have $x_i \wedge y_i = y_i \wedge x_i$ for all $i \in I$. That is, $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$ for all $i \in I$. Thus

$$\begin{aligned}
(x_i)_{i\in I} \vee (y_i)_{i\in I} &= (x_i)_{i\in I} \boxtimes ((x_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \\
&= (x_i)_{i\in I} \boxtimes (y_i *_i x_i)_{i\in I} \\
&= ((y_i *_i x_i) *_i x_i)_{i\in I} \\
&= ((x_i *_i y_i) *_i y_i)_{i\in I} \\
&= (y_i)_{i\in I} \boxtimes (x_i *_i y_i)_{i\in I} \\
&= (y_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes (x_i)_{i\in I}) \\
&= (y_i)_{i\in I} \vee (x_i)_{i\in I}.
\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is join-commutative.

Conversely, assume that $\prod_{i\in I} X_i$ is join-commutative. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Since $\prod_{i\in I} X_i$ is join-commutative, we have $f_{x_i} \vee f_{y_i} = f_{y_i} \vee f_{x_i}$. That is, $f_{x_i} \boxtimes (f_{x_i} \boxtimes f_{y_i}) = f_{y_i} \boxtimes (f_{y_i} \boxtimes f_{x_i})$. By Lemma 3.2.2, we have $f_{(y_i *_i x_i) *_i x_i} = f_{(x_i *_i y_i) *_i y_i}$. By (3.1.1), we have $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$. That is, $x_i \wedge y_i = y_i \wedge x_i$. Hence, X_i is meet-commutative for all $i \in I$. □

Next, we introduce the concept of the weak direct product of an infinite family of dBCC-algebras and obtain some of its properties as follows:

Definition 3.2.6 Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Define the *weak direct product dBCC-algebra* induced by X_i for all $i \in I$ to be the structure

$\prod_{i\in I}^{w} X_i = (\prod_{i\in I}^{w} X_i; \boxtimes)$, where

$$\prod_{i\in I}^{w} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod_{i\in I}^{w} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 3.2.7 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod_{i\in I}^{w} X_i$ is a dBCC-subalgebra of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod_{i\in I}^{w} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I}^{w} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} 0_j *_j x_j & \text{if } j \in I_1 - I_2 \\ y_j *_j x_j & \text{if } j \in I_1 \cap I_2 \\ y_j *_j 0_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (BCC-2) and (BCC-3), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} x_j & \text{if } j \in I_1 - I_2 \\ y_j *_j x_j & \text{if } j \in I_1 \cap I_2 \\ 0_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1|$, that is, it is finite. Thus $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} \in \prod_{i\in I}^{w} X_i$. Hence, $\prod_{i\in I}^{w} X_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 3.2.8 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-subalgebra of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (2.0.2), we have $x_i *_i y_i \in S_i$ for all $i \in I$, so $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By (2.0.2) and Lemma 3.2.2, we have $f_{x_i *_i y_i} = f_{y_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a BCC-subalgebra of X_i for all $i \in I$. □

Theorem 3.2.9 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a near BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a near dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a near BCC-filter of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 3.1.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i \in S_i$ for all $i \in I$, it follows from (2.0.3) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a near dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a near dBCC-filter of $\prod_{i\in I} X_i$. Since

$\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 3.1.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By (2.0.11) and by Lemma 3.2.2, we have $f_{x_i *_i y_i} = f_{y_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a near BCC-filter of X_i for all $i \in I$. $\square$

Theorem 3.2.10 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (2.0.5) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $f_{y_i} \boxtimes f_{x_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (2.0.13), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $y_i \in S_i$. Hence, S_i is a BCC-filter of X_i for all $i \in I$. $\square$

Theorem 3.2.11 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an implicative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an implicative dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an implicative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$

be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (2.0.6) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an implicative dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an implicative dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \boxtimes f_{x_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (2.0.14) and Lemma 3.2.2, we have $f_{x_i *_i z_i} = f_{z_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is an implicative BCC-filter of X_i for all $i \in I$. □

Theorem 3.2.12 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a comparative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a comparative dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a comparative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((y_i)_{i\in I} \boxtimes ((z_i)_{i\in I} \boxtimes (y_i)_{i\in I})) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i ((y_i *_i z_i) *_i y_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$, it follows from (2.0.7) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a comparative dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a comparative dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$

and $f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $(f_{y_i} \boxtimes (f_{z_i} \boxtimes f_{y_i})) \boxtimes f_{x_i} = f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$. By (2.0.15), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $y_i \in S_i$. Hence, S_i is a comparative BCC-filter of X_i for all $i \in I$. □

Theorem 3.2.13 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a shift BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a shift dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a shift BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$, it follows from (2.0.8) that $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes (z_i)_{i\in I})) = (((z_i *_i y_i) *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a shift dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a shift dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (2.0.16) and Lemma 3.2.2, we have $f_{((z_i *_i y_i) *_i y_i) *_i z_i} = f_{z_i} \boxtimes (f_{y_i} \boxtimes (f_{y_i} \boxtimes f_{z_i})) \in \prod_{i\in I} S_i$. By (3.1.1), we have $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$. Hence, S_i is a shift BCC-filter of X_i for all $i \in I$. □

Theorem 3.2.14 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-ideal of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$, it follows from (2.0.9) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (2.0.17) and Lemma 3.2.2, we have $f_{x_i *_i z_i} = f_{z_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is a BCC-ideal of X_i for all $i \in I$. □

Theorem 3.2.15 *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong dBCC-ideal of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a strong BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((x_i)_{i\in I} \boxtimes (z_i)_{i\in I}) \boxtimes ((y_i)_{i\in I} \boxtimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((z_i *_i y_i) *_i (z_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$, it follows from (2.0.10) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong dBCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong dBCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$

and $f_{(z_i*_iy_i)*_i(z_i*_ix_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (3.1.1). By Lemma 3.2.2, we have $(f_{x_i} \boxtimes f_{z_i}) \boxtimes (f_{y_i} \boxtimes f_{z_i}) = f_{(z_i*_iy_i)*_i(z_i*_ix_i)} \in \prod_{i\in I} S_i$. By (2.0.18), we have $f_{x_i} \in \prod_{i\in I} S_i$. By (3.1.1), we have $x_i \in S_i$. Hence, S_i is a strong BCC-ideal of X_i for all $i \in I$. □

For the function ψ in Definition 3.1.20, we discuss the BCC-homomorphism theorem in view of the external direct product of dBCC-algebras.

Theorem 3.2.16 *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) ψ_i is a BCC-homomorphism for all $i \in I$ if and only if ψ is a dBCC-homomorphism which is defined in Definition 3.1.20,

(ii) ψ_i is a BCC-monomorphism for all $i \in I$ if and only if ψ is a dBCC-monomorphism,

(iii) ψ_i is a BCC-epimorphism for all $i \in I$ if and only if ψ is a dBCC-epimorphism,

(iv) ψ_i is a BCC-isomorphism for all $i \in I$ if and only if ψ is a dBCC-isomorphism,

(v) $\ker\psi = \prod_{i\in I} \ker\psi_i$ and $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.

Proof. (i) Assume that ψ_i is a BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}$, $(x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) &= \psi(x'_i *_i x_i)_{i\in I}\\ &= (\psi_i(x'_i *_i x_i))_{i\in I}\\ &= (\psi_i(x'_i) \circ_i \psi_i(x_i))_{i\in I}\\ &= (\psi_i(x_i))_{i\in I} \boxtimes (\psi_i(x'_i))_{i\in I}\end{aligned}$$

$$= \psi(x_i)_{i\in I} \boxtimes \psi(x_i')_{i\in I}.$$

Hence, ψ is a dBCC-homomorphism.

Conversely, assume that ψ is a dBCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (3.1.1). Since ψ is a dBCC-homomorphism, we have $\psi(f_{x_i} \boxtimes f_{y_i}) = \psi(f_{x_i}) \boxtimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{3.2.1}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \boxtimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{3.2.2}$$

By (3.2.1) and (3.2.2), we have $\psi_i(y_i *_i x_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is a BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 3.1.21 (i).

(iii) It is straightforward from (i) and Theorem 3.1.21 (ii).

(iv) It is straightforward from (i) and Theorem 3.1.21 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I} \\
&\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I} \\
&\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I \\
&\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I \\
&\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.
\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}
(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I} \\
&\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I} \\
&\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I \\
&\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).
\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

Further, we discuss the anti-BCC-homomorphism theorem in view of the external direct product of dBCC-algebras.

Theorem 3.2.17 *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and*

$\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then

(i) ψ_i is an anti-BCC-homomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-homomorphism which is defined in Definition 3.1.20,

(ii) ψ_i is an anti-BCC-monomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-monomorphism,

(iii) ψ_i is an anti-BCC-epimorphism for all $i \in I$ if and only if ψ is an anti-dBCC-epimorphism,

(iv) ψ_i is an anti-BCC-isomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-isomorphism.

Proof. (i) Assume that ψ_i is an anti-BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) &= \psi(x'_i *_i x_i)_{i\in I} \\
&= (\psi_i(x'_i *_i x_i))_{i\in I} \\
&= (\psi_i(x_i) \circ_i \psi_i(x'_i))_{i\in I} \\
&= (\psi_i(x'_i))_{i\in I} \boxtimes (\psi_i(x_i))_{i\in I} \\
&= \psi(x'_i)_{i\in I} \boxtimes \psi(x_i)_{i\in I}.
\end{aligned}$$

Hence, ψ is an anti-dBCC-homomorphism.

Conversely, assume that ψ is an anti-dBCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (3.1.1). Since ψ is an anti-dBCC-homomorphism, we have $\psi(f_{x_i} \boxtimes f_{y_i}) = \psi(f_{y_i}) \boxtimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{3.2.3}$$

Since

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{y_i}) \boxtimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{3.2.4}$$

By (3.2.3) and (3.2.4), we have $\psi_i(y_i *_i x_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is an anti-BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 3.1.21 (i).

(iii) It is straightforward from (i) and Theorem 3.1.21 (ii).

(iv) It is straightforward from (i) and Theorem 3.1.21 (iii). $\square$

3.3 Internal direct products of BCC-algebras

Internal direct products of algebras

Before we begin the study, let's review the concept of homomorphism in algebras as follows: let $A = (A; *_A)$ and $B = (B; *_B)$ be algebras. A map $\varphi : A \to B$ is called a *homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

A (anti-)homomorphism φ is called a (anti-)monomorphism, (anti-)epimorphism, or (anti-)isomorphism if φ is injective, surjective, or bijective, respectively. If $\varphi : A \to A$ is an (anti-)isomorphism, then it is called an *(anti-)automorphism*.

Next, we will review the definition of the internal direct product of algebras and theorems introduced by Pledger [33] in 1999.

Definition 3.3.1 An algebra $(X; *)$ is called the *internal direct product* of its subalgebras X_1 and X_2 if the mapping

$$\theta : (x_1, x_2) \mapsto x_1 * x_2 \tag{3.3.1}$$

is an isomorphism from the algebra $(X_1 \times X_2; \otimes)$ to X.

Then $\theta^{-1} : X \to X_1 \times X_2$ is an isomorphism. Let $\alpha_1 : X \to X_1$ and $\alpha_2 : X \to X_2$ be such that

$$(\forall x \in X)(\theta^{-1}(x) = (\alpha_1(x), \alpha_2(x))). \tag{3.3.2}$$

Lemma 3.3.2 *Let an algebra $(X;*)$ be the internal direct product of its subalgebras X_1 and X_2. Then*

(i) $\alpha_1(X) = X_1$.

(ii) $\alpha_2(X) = X_2$.

We conclude that α_1 and α_2 are surjective.

Theorem 3.3.3 *Let an algebra $(X;*)$ be the internal direct product of its subalgebras X_1 and X_2. Then $\forall x_1, y_1 \in X_1, \forall x_2, y_2 \in X_2, (x_1 * x_2) * (y_1 * y_2) = (x_1 * y_1) * (x_2 * y_2)$.*

Theorem 3.3.4 *Let $(X; *, \alpha_1, \alpha_2)$ be an algebra of type $(2, 1, 1)$. Then the algebra $(X;*)$ is the internal direct product of $\alpha_1(X)$ and $\alpha_2(X)$ if and only if the algebra $(X; *, \alpha_1, \alpha_2)$ has the following properties:*

(i) $\forall x \in X, \alpha_1(x) * \alpha_2(x) = x$,

(ii) $\forall x_1, x_2 \in X, \alpha_1(x_1) = \alpha_1(\alpha_1(x_1) * \alpha_2(x_2))$ *and* $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1) * \alpha_2(x_2))$, *in particular,* $\forall x_1 \in X_1, \forall x_2 \in X_2$, $\alpha_1(x_1 * x_2) = x_1$ *and* $\alpha_2(x_1 * x_2) = x_2$,

(iii) α_1 *and* α_2 *are homomorphisms. Moreover,* $\alpha_1(X_1), \alpha_1(X_2), \alpha_2(X_1)$ *and* $\alpha_2(X_2)$ *are subalgebras of* X.

Definition 3.3.5 Let $(X;*)$ be an algebra. For any $a \in X$,

$$(\forall x \in X)(\rho_a(x) = a * x), \tag{3.3.3}$$

and

$$(\forall x \in X)(\lambda_a(x) = x * a). \tag{3.3.4}$$

Theorem 3.3.6 *In any internal direct product* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X), \alpha_1 |_{X_1}$ *and* $\alpha_2 |_{X_2}$ *are injections.*

Corollary 3.3.7 *In any internal direct product* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, for any* $a \in X_2, \rho_{\alpha_1(a)} |_{\alpha_1(X_1)}$ *is a left inverse of* $\alpha_1 |_{X_1}$*. For any* $a \in X_1, \lambda_{\alpha_2(a)} |_{\alpha_2(X_2)}$ *is left inverse of* $\alpha_2 |_{X_2}$*.*

Theorem 3.3.8 *In any internal direct product* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, the following conditions are equivalent.*

(*i*) $\alpha_1(X_1) = X_1$,

(*ii*) *there exists* $d_1 \in X_1$ *such that* $\alpha_1(X_2) = \{d_1\}$ *with* $\alpha_1(d_1) = d_1$,

(*iii*) $\alpha_1(X_1) \cap \alpha_1(X_2) = \{d_1\}$,

(*iv*) $\alpha_1(X_1) \cap \alpha_1(X_2) \neq \emptyset$.

Corollary 3.3.9 *Under the conditions of Theorem 3.3.8* (*i*)-(*iv*)*,* $\rho_{d_1} |_{X_1}$ *is the inverse automorphism of* $\alpha_1 |_{X_1}$*. Under the corresponding conditions with transposed subscripts,* $\lambda_{d_2} |_{X_2}$ *is the inverse automorphism of* $\alpha_2 |_{X_2}$*.*

Theorem 3.3.10 *In any internal direct product* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, the following conditions are equivalent.*

(*i*) $\alpha_1(X_1) = X_1$ *and* $\alpha_2(X_2) = X_2$,

(*ii*) *there exists* $d \in X$ *such that* $\alpha_1(X_2) = \alpha_2(X_1) = \{d\}$ *with* $\alpha_1(d) = \alpha_2(d) = d$ *and* $d * d = d$,

(*iii*) $X_1 \cap X_2 = \{d\}$,

(*iv*) $X_1 \cap X_2 \neq \emptyset$.

Corollary 3.3.11 *Under the conditions of Theorem 3.3.10 (i)-(iv),* $\rho_d \mid_{X_1}$ *is the inverse automorphisms of* $\alpha_1 \mid_{X_1}$ *and* $\lambda_d \mid_{X_2}$ *is the inverse automorphisms* $\alpha_2 \mid_{X_2}$ *with* $d * d = d$.

Corollary 3.3.12 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, $\mid X_1 \cap X_2 \mid$ *can only be* 1 *or* 0.

Theorem 3.3.13 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *if* $X_1 \cap X_2 = \{d\}$, *and* d *commutes with every element of* X, *then every element of* X_1 *commutes with every element of* X_2.

Theorem 3.3.14 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$, $X_2 = \alpha_2(X)$ *and* $X_1 \cap X_2 = \{d\}$, $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$ *and* $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$ *(i.e.,* $\alpha_1^2 = \alpha_1$ *and* $\alpha_2^2 = \alpha_2$*) if and only if* d *is both a right identity element for* X_1 *and a left identity element for* X_2.

Theorem 3.3.15 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *if every element of* X_1 *commutes with every element of* X_2, *and* $\alpha_1^2 = \alpha_1$ *and* $\alpha_2^2 = \alpha_2$, *then* X *has an identity element.*

Theorem 3.3.16 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *if* X *is finite, then* $\mid X_1 \cap X_2 \mid = 1$.

Theorem 3.3.17 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *if* X *satisfies either the right or left cancellation law, then* $\mid X_1 \cap X_2 \mid = 1$.

Theorem 3.3.18 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$, $X_2 = \alpha_2(X)$ *and* $X_1 \cap X_2 = \{d\}$, $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$ *and* $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$ *(i.e.,* $\alpha_1^2 = \alpha_1$ *and* $\alpha_2^2 = \alpha_2$*) if and only if* $\forall x_1 \in X_1, \forall y \in X, \forall z_2 \in X_2$, $(x_1 * y) * z_2 = x_1 * (y * z_2)$.

Corollary 3.3.19 *In any internal direct product* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$, $X_2 = \alpha_2(X)$, *and* $X_1 \cap X_2 = \{d\}$, *if* X *is a semigroup (An algebra* $X = (X; *)$ *of* *type 2 is called a semigroup if its operation $*$ satisfies the associative law), then $\alpha_1 |_{X_1} = \boldsymbol{1}_{X_1}$ and $\alpha_2 |_{X_2} = \boldsymbol{1}_{X_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) with d is the right identity element for X_1 and the left identity element for X_2.*

Applying internal direct products to BCC-algebras

We apply the results from the internal direct product of algebras to the internal direct product of BCC-algebras and get the following results.

Using Theorem 3.3.3, (BCC-2), (BCC-3), and (2.0.1), we get the following theorem.

Theorem 3.3.20 *Let a BCC-algebra $(X; *, 0)$ be the internal direct product of its BCC-subalgebras X_1 and X_2. Then*

(*i*) $\forall x_1 \in X_1, \forall x_2, y_2 \in X_2, (x_1 * x_2) * y_2 = x_2 * y_2,$

(*ii*) $\forall y_1 \in X_1, \forall x_2, y_2 \in X_2, x_2 * (y_1 * y_2) = y_1 * (x_2 * y_2),$

(*iii*) $\forall x_1, y_1 \in X_1, \forall y_2 \in X_2, y_1 * y_2 = (x_1 * y_1) * y_2,$

(*iv*) $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall y \in X_1 \cap X_2, 0 = (x_1 * y) * (x_2 * y),$

(*v*) $\forall y_1 \in X_1, \forall y_2 \in X_2, \forall x \in X_1 \cap X_2, y_1 * y_2 = (x * y_1) * (x * y_2).$

Theorem 3.3.21 *Let $(X; *, \alpha_1, \alpha_2, 0)$ be a BCC-algebra and unary operations α_1 and α_2. The algebra $(X; *)$ is the internal direct product of $\alpha_1(X)$ and $\alpha_2(X)$ if and only if*

(*i*) $\alpha_1 = \boldsymbol{0}_X$ *is the zero function,*

(*ii*) $\alpha_2 = \boldsymbol{1}_X$ *is the identity function.*

Proof. Assume that $(X; *, \alpha_1, \alpha_2, 0)$ is a BCC-algebra and unary operations α_1 and α_2. The algebra $(X; *)$ is the internal direct product of $\alpha_1(X)$ and $\alpha_2(X)$. Then

$$\begin{aligned} \alpha_2(0) &= \alpha_2(0 * 0) && (2.0.1) \\ &= \alpha_2(0) * \alpha_2(0) && (\text{Theorem 3.3.4 } (iii)) \\ &= 0 && (2.0.1) \end{aligned}$$

and

$$\begin{aligned} \alpha_1(0) &= \alpha_1(0 * 0) && (2.0.1) \\ &= \alpha_1(0) * \alpha_1(0) && (\text{Theorem 3.3.4 } (iii)) \\ &= 0. && (2.0.1) \end{aligned}$$

(i) Let $x \in X$. Then

$$\begin{aligned} \alpha_1(x) &= \alpha_1(\alpha_1(x) * \alpha_2(0)) && (\text{Theorem 3.3.4 } (ii)) \\ &= \alpha_1(\alpha_1(x) * 0) \\ &= \alpha_1(0) && (\text{BCC-3}) \\ &= 0. \end{aligned}$$

Hence, $\alpha_1 = \mathbf{0}_X$.

(ii) Let $x \in X$. Then

$$\begin{aligned} x &= \alpha_1(x) * \alpha_2(x) && (\text{Theorem 3.3.4 } (i)) \\ &= 0 * \alpha_2(x) && ((i)) \\ &= \alpha_2(x). && (\text{BCC-2}) \end{aligned}$$

Hence, $\alpha_2 = \mathbf{1}_X$.

Conversely, assume that $\alpha_1 = \mathbf{0}_X$ and $\alpha_2 = \mathbf{1}_X$. Then (i)-(iii) in Theorem 3.3.4 hold. Hence, $(X;*)$ is the internal direct product of $\alpha_1(X) = \{0\}$ and $\alpha_2(X) = X$. □

By Theorem 3.3.21, we have the following theorem.

Theorem 3.3.22 *Every BCC-algebra* $(X;*,0)$ *is only the internal direct product of* $\{0\}$ *and* X*.*

3.4 Anti-internal direct products of BCC-algebras

Anti-internal direct products of algebras

In this section, we introduce the concept of the anti-internal direct product of algebras and find important theorems.

Definition 3.4.1 An algebra $(X;*)$ is called the *anti-internal direct product* of its subalgebras X_1 and X_2 if the mapping

$$\phi : (x_1, x_2) \mapsto x_2 * x_1 \tag{3.4.1}$$

is an isomorphism from the algebra $(X_1 \times X_2; \otimes)$ to X.

Then $\phi^{-1} : X \to X_1 \times X_2$ is an isomorphism. Let $\beta_1 : X \to X_1$ and $\beta_2 : X \to X_2$ be such that

$$(\forall x \in X)(\phi^{-1}(x) = (\beta_1(x), \beta_2(x))). \tag{3.4.2}$$

Lemma 3.4.2 *Let an algebra* $(X;*)$ *be the anti-internal direct product of its subalgebras* X_1 *and* X_2*. Then* (*i*) $\beta_1(X) = X_1$.

(*ii*) $\beta_2(X) = X_2$.

We conclude that β_1 and β_2 are surjective.

Proof. (*i*) Clearly, $\beta_1(X) \subseteq X_1$. Let $x_1 \in X_1$ and choose $x_2 \in X_2$. Since ϕ^{-1} is surjective, there exists $x \in X$ such that

$$\begin{aligned}(x_1, x_2) &= \phi^{-1}(x) \\ &= (\beta_1(x), \beta_2(x)). \qquad ((3.4.2))\end{aligned}$$

Thus $x_1 = \beta_1(x) \in \beta_1(X)$, that is, $X_1 \subseteq \beta_1(X)$. Hence, $\beta_1(X) = X_1$.

(*ii*) Clearly, $\beta_2(X) \subseteq X_2$. Let $x_2 \in X_2$ and choose $x_1 \in X_1$. Since ϕ^{-1} is surjective, there exists $x \in X$ such that

$$\begin{aligned}(x_1, x_2) &= \phi^{-1}(x) \\ &= (\beta_1(x), \beta_2(x)). \qquad ((3.4.2))\end{aligned}$$

Thus $x_2 = \beta_2(x) \in \beta_2(X)$, that is, $X_2 \subseteq \beta_2(X)$. Hence, $\beta_2(X) = X_2$. □

Theorem 3.4.3 *Let an algebra $(X; *)$ be the anti-internal direct product of its subalgebras X_1 and X_2. Then $\forall x_1, y_1 \in X_1, \forall x_2, y_2 \in X_2, (x_2 * x_1) * (y_2 * y_1) = (x_2 * y_2) * (x_1 * y_1)$.*

Proof. Let $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Then

$$\begin{aligned}(x_2 * x_1) * (y_2 * y_1) &= \phi(x_1, x_2) * \phi(y_1, y_2) && ((3.4.1)) \\ &= \phi((x_1, x_2) \otimes (y_1, y_2)) && \text{(Homomorphism)} \\ &= \phi(x_1 * y_1, x_2 * y_2) && \text{(Definition 3.1.1)}\end{aligned}$$

$$= (x_2 * y_2) * (x_1 * y_1). \qquad ((3.4.1))$$

□

Theorem 3.4.4 *Let* $(X; *, \beta_1, \beta_2)$ *be an algebra of type* $(2, 1, 1)$. *Then the algebra* $(X; *)$ *is the anti-internal direct product of* $\beta_1(X)$ *and* $\beta_2(X)$ *if and only if the algebra* $(X; *, \beta_1, \beta_2)$ *has the following properties:*

(*i*) $\forall x \in X, \beta_2(x) * \beta_1(x) = x$,

(*ii*) $\forall x_1, x_2 \in X, \beta_1(x_1) = \beta_1(\beta_2(x_2) * \beta_1(x_1))$ *and* $\beta_2(x_2) = \beta_2(\beta_2(x_2) * \beta_1(x_1))$, *in particular,* $\forall x_1 \in X_1, \forall x_2 \in X_2$, $\beta_1(x_2 * x_1) = x_1$ *and* $\beta_2(x_2 * x_1) = x_2$,

(*iii*) β_1 *and* β_2 *are homomorphisms. Moreover,* $\beta_1(X_1), \beta_1(X_2), \beta_2(X_1)$ *and* $\beta_2(X_2)$ *are subalgebras of* X.

Proof. Write $\beta_1(X) = X_1$ and $\beta_2(X) = X_2$. First, assume $(X; *)$ is the anti-internal direct product of X_1 and X_2.

(*i*) Let $x \in X$. Then

$$\begin{aligned} x &= \phi(\phi^{-1}(x)) \\ &= \phi(\beta_1(x), \beta_2(x)) && ((3.4.2)) \\ &= \beta_2(x) * \beta_1(x). && ((3.4.1)) \end{aligned}$$

(*ii*) Let $x_1, x_2 \in X$. Then there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $\beta_1(x_1) = y_1$ and $\beta_2(x_2) = y_2$. Thus

$$\begin{aligned} (\beta_1(x_1), \beta_2(x_2)) &= (y_1, y_2) \\ &= \phi^{-1}(\phi(y_1, y_2)) \\ &= \phi^{-1}(y_2 * y_1) && ((3.4.1)) \end{aligned}$$

$$= (\beta_1(y_2 * y_1), \beta_2(y_2 * y_1)) \quad ((3.4.2))$$
$$= (\beta_1(\beta_2(x_2) * \beta_1(x_1)), \beta_2(\beta_2(x_2) * \beta_1(x_1))).$$

Hence, $\beta_1(x_1) = \beta_1(\beta_2(x_2) * \beta_1(x_1))$ and $\beta_2(x_2) = \beta_2(\beta_2(x_2) * \beta_1(x_1))$.

(iii) Let $x, y \in X$. Then

$$\begin{aligned}
(\beta_1(x * y), \beta_2(x * y)) &= \phi^{-1}(x * y) && ((3.4.2)) \\
&= \phi^{-1}(x) \otimes \phi^{-1}(y) && \text{(Homomorphism)} \\
&= (\beta_1(x), \beta_2(x)) \otimes (\beta_1(y), \beta_2(y)) && ((3.4.2)) \\
&= (\beta_1(x) * \beta_1(y), \beta_2(x) * \beta_2(y)). && \text{(Definition 3.1.1)}
\end{aligned}$$

Hence, $\beta_1(x * y) = \beta_1(x) * \beta_1(y)$ and $\beta_2(x * y) = \beta_2(x) * \beta_2(y)$. Therefore β_1 and β_2 are homomorphisms.

Conversely, assume (i)-(iii) are satisfied. By (iii), we have $\beta_1(X) = X_1$ and $\beta_2(X) = X_2$ are subalgebras of X. Define the function $\eta : X \to X_1 \times X_2$ by

$$(\forall x \in X)(\eta(x) = (\beta_1(x), \beta_2(x))). \quad (3.4.3)$$

Let $x, y \in X$ be such that $\eta(x) = \eta(y)$. Then

$$\begin{aligned}
\eta(x) = \eta(y) &\Rightarrow \beta_1(x) = \beta_1(y) \text{ and } \beta_2(x) = \beta_2(y) \\
&\Rightarrow \beta_2(x) * \beta_1(x) = \beta_2(y) * \beta_1(y) \\
&\Rightarrow x = y. && ((i))
\end{aligned}$$

So, η is injective.

Also, let $(x_1, x_2) \in X_1 \times X_2$. Then $x_2 * x_1 \in X$ such that

$$\begin{aligned} \eta(x_2 * x_1) &= (\beta_1(x_2 * x_1), \beta_2(x_2 * x_1)) && ((3.4.3)) \\ &= (x_1, x_2). && ((ii)) \end{aligned}$$

So, η is surjective.

Also, let $x, y \in X$. Then

$$\begin{aligned} \eta(x) \otimes \eta(y) &= (\beta_1(x), \beta_2(x)) \otimes (\beta_1(y), \beta_2(y)) && ((3.4.3)) \\ &= (\beta_1(x) * \beta_1(y), \beta_2(x) * \beta_2(y)) && \text{(Definition 3.1.1)} \\ &= (\beta_1(x * y), \beta_2(x * y)) && ((iii)) \\ &= \eta(x * y). && ((3.4.3)) \end{aligned}$$

So, η is a homomorphism. Hence, η is an isomorphism and so η^{-1} is an isomorphism.

Finally let $\phi = \eta^{-1}$. Then X is an anti-internal direct product of X_1 and X_2. □

Theorem 3.4.5 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$, $\beta_1 |_{X_1}$ *and* $\beta_2 |_{X_2}$ *are injective.*

Proof. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$\begin{aligned} (\rho_{\beta_1(a)} \circ \beta_1 |_{X_1})(x) &= \rho_{\beta_1(a)}(\beta_1 |_{X_1} (x)) \\ &= \rho_{\beta_1(a)}(\beta_1(x)) \\ &= \beta_1(a) * \beta_1(x) && ((3.3.3)) \\ &= \beta_1(a * x) && \text{(Theorem 3.4.4 } (iii)) \\ &= x && \text{(Theorem 3.4.4 } (ii)) \end{aligned}$$

$$= \mathbf{1}_{X_1}(x).$$

Thus $\beta_1 \mid_{X_1}$ is injective.

And, choose any $a \in X_1$. Let $x \in X_2$. Then

$$\begin{aligned}
(\lambda_{\beta_2(a)} \circ \beta_2 \mid_{X_2})(x) &= \lambda_{\beta_2(a)}(\beta_2 \mid_{X_2} (x)) \\
&= \lambda_{\beta_2(a)}(\beta_2(x)) \\
&= \beta_2(x) * \beta_2(a) && ((3.3.4)) \\
&= \beta_2(x * a) && (\text{Theorem 3.4.4 } (iii)) \\
&= x && (\text{Theorem 3.4.4 } (ii)) \\
&= \mathbf{1}_{X_2}(x).
\end{aligned}$$

Thus $\beta_2 \mid_{X_2}$ is injective. □

Corollary 3.4.6 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, for any* $a \in X_2, \rho_{\beta_1(a)} \mid_{\beta_1(X_1)}$ *is a left inverse of* $\beta_1 \mid_{X_1}$*. For any* $a \in X_1, \lambda_{\beta_2(a)} \mid_{\beta_2(X_2)}$ *is a left inverse of* $\beta_2 \mid_{X_2}$*.*

Theorem 3.4.7 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, the following conditions are equivalent.*

(*i*) $\beta_1(X_1) = X_1$,

(*ii*) *there exists* $d_1 \in X_1$ *such that* $\beta_1(X_2) = \{d_1\}$ *with* $\beta_1(d_1) = d_1$,

(*iii*) $\beta_1(X_1) \cap \beta_1(X_2) = \{d_1\}$,

(*iv*) $\beta_1(X_1) \cap \beta_1(X_2) \neq \emptyset$.

Proof. (*i*)⇒(*ii*) Choose any $c_1 \in X_1$. Then $\beta_1(\beta_2(c_1)) \in \beta_1(X) = X_1 = \beta_1(X_1)$ from (*i*), so there exists $d_1 \in X_1$ such that $\beta_1(\beta_2(c_1)) = \beta_1(d_1)$. So, let $x_2 \in X_2$.

Then

$$\begin{aligned}
\beta_1(x_2) &= \beta_1(\beta_2(x_2 * c_1)) && \text{(Theorem 3.4.4 } (ii))\\
&= \beta_1(\beta_2(x_2) * \beta_2(c_1)) && \text{(Theorem 3.4.4 } (iii))\\
&= \beta_1(\beta_2(x_2)) * \beta_1(\beta_2(c_1)) && \text{(Theorem 3.4.4 } (iii))\\
&= \beta_1(\beta_2(x_2)) * \beta_1(d_1) && (\beta_1(\beta_2(c_1)) = \beta_1(d_1))\\
&= \beta_1(\beta_2(x_2) * d_1) && \text{(Theorem 3.4.4 } (iii))\\
&= d_1. && \text{(Theorem 3.4.4 } (ii))
\end{aligned}$$

So, $\beta_1(X_2) \subseteq \{d_1\}$. Since $\beta_1(X_2)$ is nonempty, we have $\beta_1(X_2) = \{d_1\}$. Also $\beta_1(d_1) = \beta_1(\beta_2(c_1)) \in \beta_1(X_2) = \{d_1\}$, so $\beta_1(d_1) = d_1$. Hence, $\beta_1(X_2)$ is a singleton $\{d_1\}$ with $\beta_1(d_1) = d_1$.

$(ii)\Rightarrow(iii)$ From (ii), we have $\beta_1(X_2) = \{d_1\} = \{\beta_1(d_1)\} \subseteq \beta_1(X_1)$. Hence, $\beta_1(X_1) \cap \beta_1(X_2)$ is the singleton $\{d_1\}$.

$(iii)\Rightarrow(iv)$ Obviously.

$(iv)\Rightarrow(i)$ From (iv), there exist $a_1 \in X_1$ and $a_2 \in X_2$ such that $\beta_1(a_1) = \beta_1(a_2)$. Clearly, $\beta_1(X_1) \subseteq X_1$. So, let $x \in X_1$. Then

$$\begin{aligned}
x &= \beta_1(a_2 * x) && \text{(Theorem 3.4.4 } (ii))\\
&= \beta_1(a_2) * \beta_1(x) && \text{(Theorem 3.4.4 } (iii))\\
&= \beta_1(a_1) * \beta_1(x) && (\beta_1(a_1) = \beta_1(a_2))\\
&= \beta_1(a_1 * x) && \text{(Theorem 3.4.4 } (iii))\\
&\in \beta_1(X_1).
\end{aligned}$$

So, $X_1 \subseteq \beta_1(X_1)$. Hence, $\beta_1(X_1) = X_1$. □

Theorem 3.4.8 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 =$

$\beta_1(X)$ and $X_2 = \beta_2(X)$, the following conditions are equivalent.

(i) $\beta_2(X_2) = X_2$,

(ii) *there exists* $d_2 \in X_2$ *such that* $\beta_2(X_1) = \{d_2\}$ *with* $\beta_2(d_2) = d_2$,

(iii) $\beta_2(X_1) \cap \beta_2(X_2) = \{d_2\}$,

(iv) $\beta_2(X_1) \cap \beta_2(X_2) \neq \emptyset$.

Proof. Prove it using the same principle as Theorem 3.4.7, replacing the subscript 1 to 2 and 2 to 1. □

Corollary 3.4.9 *Under the conditions of Theorem 3.4.7 (i)-(iv), $\rho_{d_1} |_{X_1}$ is the inverse automorphism of $\beta_1 |_{X_1}$. Moreover, $\forall a \in X_2$, $\rho_{d_1} |_{X_1} = \rho_a |_{X_1}$.*

Proof. We shall prove that $\rho_{d_1} |_{X_1}$ is the left inverse automorphism of $\beta_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$\begin{aligned}
(\rho_{d_1} |_{X_1} \circ \beta_1 |_{X_1})(x) &= \rho_{d_1} |_{X_1} (\beta_1 |_{X_1} (x)) \\
&= \rho_{d_1}(\beta_1(x)) \\
&= \rho_{\beta_1(a)}(\beta_1(x)) && \text{(Theorem 3.4.7 } (ii)) \\
&= \beta_1(a) * \beta_1(x) && ((3.3.3)) \\
&= \beta_1(a * x) && \text{(Theorem 3.4.4 } (iii)) \\
&= x. && \text{(Theorem 3.4.4 } (ii))
\end{aligned}$$

Thus $\beta_1 |_{X_1}$ is injective.

Next, we shall prove that $\rho_a |_{X_1}$ is the right inverse automorphism of $\beta_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$(\beta_1 |_{X_1} \circ \rho_a |_{X_1})(x) = \beta_1 |_{X_1} (\rho_a(x))$$

$$= \beta_1(\rho_a(x))$$

$$= \beta_1(a * x) \qquad ((3.3.3))$$

$$= x. \qquad \text{(Theorem 3.4.4 } (ii))$$

Thus $\beta_1 |_{X_1}$ is surjective, so $\beta_1 |_{X_1}$ is bijective. Hence, $\rho_{d_1} |_{X_1} = \rho_a |_{X_1}$ is the inverse of $\beta_1 |_{X_1}$. By Theorem 3.4.4 (iii), we have $\beta_1 |_{X_1}$ is an automorphism. Therefore, $\rho_{d_1} |_{X_1} = \rho_a |_{X_1}$ is the inverse automorphism of $\beta_1 |_{X_1}$. □

Corollary 3.4.10 *Under the conditions of Theorem 3.4.8* (i)-(iv)*,* $\lambda_{d_2} |_{X_2}$ *is the inverse automorphism of* $\beta_2 |_{X_2}$*. Moreover,* $\forall a \in X_1$*,* $\lambda_{d_2} |_{X_2} = \lambda_a |_{X_2}$*.*

Proof. The proof is in the same way as Corollary 3.4.9. □

Theorem 3.4.11 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, the following conditions are equivalent.*

(i) $\beta_1(X_1) = X_1$ *and* $\beta_2(X_2) = X_2$*,*

(ii) *there exists* $d \in X$ *such that* $\beta_1(X_2) = \beta_2(X_1) = \{d\}$ *with* $\beta_1(d) = \beta_2(d) = d$ *and* $d * d = d$*,*

(iii) $X_1 \cap X_2 = \{d\}$*,*

(iv) $X_1 \cap X_2 \neq \emptyset$*.*

Proof. $(i) \Rightarrow (ii)$ From (i), Theorem 3.4.7 gives $\beta_1(X_2) = \{d_1\}$ with $\beta_1(d_1) = d_1$ for some $d_1 \in X_1$ and Theorem 3.4.8 gives $\beta_2(X_1) = \{d_2\}$ with $\beta_2(d_2) = d_2$ for some $d_2 \in X_2$. Also, $\beta_1(d_2) \in \beta_1(X_2) = \{d_1\}$ and $\beta_2(d_1) \in \beta_2(X_1) = \{d_2\}$, so $\beta_1(d_2) = d_1$ and $\beta_2(d_1) = d_2$. Thus

$$d_1 = \beta_2(d_1) * \beta_1(d_1) \qquad \text{(Theorem 3.4.4 } (i))$$

$$= d_2 * d_1 \qquad (\beta_2(d_1) = d_2, \beta_1(d_1) = d_1)$$

$$= \beta_2(d_2) * \beta_1(d_2) \qquad (\beta_2(d_2) = d_2, \beta_1(d_2) = d_1)$$
$$= d_2. \qquad \text{(Theorem 3.4.4 } (i))$$

Choose $d = d_1$. Hence, $\beta_1(X_2) = \beta_2(X_1) = \{d\}$ with $\beta_1(d) = \beta_2(d) = d$ and $d * d = d$.

$(ii) \Rightarrow (iii)$ From (ii), $d \in \beta_1(X_2) \cap \beta_2(X_1) \subseteq X_1 \cap X_2$. Thus $\{d\} \subseteq X_1 \cap X_2$. Let $x \in X_1 \cap X_2$. Then $\beta_1(x) \in \beta_1(X_1 \cap X_2) \subseteq \beta_1(X_2) = \{d\}$ and $\beta_2(x) \in \beta_2(X_1 \cap X_2) \subseteq \beta_2(X_1) = \{d\}$. Thus

$$x = \beta_2(x) * \beta_1(x) \qquad \text{(Theorem 3.4.4 } (i))$$
$$= d * d$$
$$= d. \qquad \text{(Assumption)}$$

Thus, $X_1 \cap X_2 \subseteq \{d\}$. Hence, $X_1 \cap X_2 = \{d\}$.

$(iii) \Rightarrow (iv)$ Obviously.

$(iv) \Rightarrow (i)$ We see that $\beta_1(X_1 \cap X_2) \subseteq \beta_1(X_1) \cap \beta_1(X_2)$. Thus

$$X_1 \cap X_2 \neq \emptyset \Rightarrow \beta_1(X_1) \cap \beta_1(X_2) \neq \emptyset$$
$$\Rightarrow \beta_1(X_1) = X_1 \text{ and } \beta_2(X_2) = X_2.$$
$$\text{(Theorems 3.4.7 } (i) \text{ and 3.4.8 } (i))$$

Hence, $\beta_1(X_1) = X_1$ and $\beta_2(X_2) = X_2$. □

Corollary 3.4.12 *Under the conditions of Theorem 3.4.11 (i)-(iv), $\rho_d |_{X_1}$ is the inverse automorphism of $\beta_1 |_{X_1}$ and $\lambda_d |_{X_2}$ is the inverse automorphism of $\beta_2 |_{X_2}$. Moreover, $\forall a \in X_2, \forall b \in X_1$, $\rho_d |_{X_1} = \rho_a |_{X_1}$ and $\lambda_d |_{X_2} = \lambda_b |_{X_2}$.*

Proof. It is a direct result of Corollaries 3.4.9 and 3.4.10. □

Corollary 3.4.13 *In any anti-internal direct product $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X)$ and $X_2 = \beta_2(X)$, $|X_1 \cap X_2|$ can only be 1 or 0.*

Proof. It follows from Theorem 3.4.11 (iii) and (iv). □

Theorem 3.4.14 *In any anti-internal direct product $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X)$ and $X_2 = \beta_2(X)$, if $X_1 \cap X_2 = \{d\}$, and d commutes with every element of X, then every element of X_1 commutes with every element of X_2.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then

$$\begin{aligned}
x_1 * x_2 &= (\beta_2(x_1) * \beta_1(x_1)) * (\beta_2(x_2) * \beta_1(x_2)) && \text{(Theorem 3.4.4 } (i)) \\
&= (d * \beta_1(x_1)) * (\beta_2(x_2) * d) && \text{(Theorem 3.4.11 } (ii)) \\
&= (d * \beta_2(x_2)) * (\beta_1(x_1) * d) && \text{(Theorem 3.4.3)} \\
&= (\beta_2(x_2) * d) * (d * \beta_1(x_1)) && \\
&= (\beta_2(x_2) * \beta_1(x_2)) * (\beta_2(x_1) * \beta_1(x_1)) && \text{(Theorem 3.4.11 } (ii)) \\
&= x_2 * x_1. && \text{(Theorem 3.4.4 } (i))
\end{aligned}$$

The proof is completed. □

Theorem 3.4.15 *In any anti-internal direct product $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X)$, $X_2 = \beta_2(X)$ and $X_1 \cap X_2 = \{d\}$, $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$ (i.e., $\beta_1^2 = \beta_1$ and $\beta_2^2 = \beta_2$) if and only if d is both the left identity element for X_1 and the right identity element for X_2.*

Proof. Assume both $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$. Then Corollary 3.4.12 gives $\rho_d |_{X_1} = \mathbf{1}_{X_1}$ and $\lambda_d |_{X_2} = \mathbf{1}_{X_2}$. Let $x_1 \in X_1$. Then $d * x_1 = \rho_d(x_1) = \rho_d |_{X_1} (x_1) = \mathbf{1}_{X_1}(x_1) = x_1$, that is, d is the left identity element for X_1. Let $x_2 \in X_2$. Then $x_2 * d = \lambda_d(x_2) = \lambda_d |_{X_2} (x_2) = \mathbf{1}_{X_2}(x_2) = x_2$, that is, d is the right identity element for X_2. Hence, d is both the left identity element for X_1 and the right identity element for X_2.

Conversely, assume that d is both the left identity element for X_1 and the right identity element for X_2. By the assumption, Theorem 3.4.11 (iii) holds. Thus by Theorem 3.4.11, we have the conditions (i)-(iv). It follows from Corollary 3.4.12 that $\rho_d |_{X_1} \circ \beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} \circ \lambda_d |_{X_2} = \mathbf{1}_{X_2}$. Let $x_1 \in X_1$. Since d is the left identity element for X_1, we have $\mathbf{1}_{X_1}(x_1) = (\rho_d |_{X_1} \circ \beta_1 |_{X_1})(x_1) = \rho_d |_{X_1} (\beta_1 |_{X_1} (x_1)) = \rho_d(\beta_1 |_{X_1} (x_1)) = d * \beta_1 |_{X_1} (x_1) = \beta_1 |_{X_1} (x_1)$. Thus $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$. And let $x_2 \in X_2$. Since d is the right identity element for X_2, we have $\mathbf{1}_{X_2}(x_2) = (\beta_2 |_{X_2} \circ \lambda_d |_{X_2})(x_2) = \beta_2 |_{X_2} (\lambda_d |_{X_2} (x_2)) = \beta_2 |_{X_2} (\lambda_d(x_2)) = \beta_2 |_{X_2} (x_2 * d) = \beta_2 |_{X_2} (x_2)$. Thus $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$. □

Theorem 3.4.16 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$, $X_2 = \beta_2(X)$ *and* $X_1 \cap X_2 = \{d\}$, *if every element of* X_1 *commutes with every element of* X_2, *and* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$, *then* X *has the identity element.*

Proof. Assume that every element of X_1 commutes with every element of X_2, and $\beta_1^2 = \beta_1$ and $\beta_2^2 = \beta_2$. By Theorem 3.4.15, we have d is the left identity element for X_1 and d is the right identity element for X_2. Let $x \in X$. Then

$$\begin{aligned}
x * d &= (\beta_2(x) * \beta_1(x)) * (d * d) && \text{(Theorem 3.4.4 } (i))\\
&= (\beta_2(x) * d) * (\beta_1(x) * d) && \text{(Theorem 3.4.3)}\\
&= \beta_2(x) * (\beta_1(x) * d) && (d \text{ is the right identity for } X_2)\\
&= \beta_2(x) * (d * \beta_1(x)) && \text{(Assumption)}\\
&= \beta_2(x) * \beta_1(x) && (d \text{ is the left identity for } X_1)\\
&= x && \text{(Theorem 3.4.4 } (i))
\end{aligned}$$

and

$$
\begin{aligned}
d * x &= (d * d) * (\beta_2(x) * \beta_1(x)) && \text{(Theorem 3.4.4 } (i)) \\
&= (d * \beta_2(x)) * (d * \beta_1(x)) && \text{(Theorem 3.4.3)} \\
&= (d * \beta_2(x)) * \beta_1(x) && (d \text{ is the left identity for } X_1) \\
&= (\beta_2(x) * d) * \beta_1(x) && \text{(Assumption)} \\
&= \beta_2(x) * \beta_1(x) && (d \text{ is the right identity for } X_2) \\
&= x. && \text{(Theorem 3.4.4 } (i))
\end{aligned}
$$

Hence, d is the identity element of X. □

Theorem 3.4.17 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, if* X_1 *and* X_2 *are finite, then* $| X_1 \cap X_2 |= 1$.

Proof. Assume that X_1 and X_2 are finite. By Theorem 3.4.5, we have $\beta_1 |_{X_1}$ and $\beta_2 |_{X_2}$ are injective. By the assumption, we have $\beta_1(X_1) = \beta_1 |_{X_1} (X_1) = X_1$ and $\beta_2(X_2) = \beta_2 |_{X_2} (X_2) = X_2$. It follows from Theorem 3.4.11 that $| X_1 \cap X_2 |= 1$. □

Theorem 3.4.18 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, if* X *satisfies either the right or left cancellation law, then* $| X_1 \cap X_2 |= 1$.

Proof. Assume that X satisfies the right cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$
\begin{aligned}
\beta_2(a_1) * \beta_1(a_1) &= a_1 && \text{(Theorem 3.4.4 } (i)) \\
&= \beta_1(a_2 * a_1) && \text{(Theorem 3.4.4 } (ii)) \\
&= \beta_1(a_2) * \beta_1(a_1). && \text{(Theorem 3.4.4 } (iii))
\end{aligned}
$$

Cancel $\beta_1(a_1)$ from the right so $\beta_2(a_1) = \beta_1(a_2)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.4.11 that $\mid X_1 \cap X_2 \mid = 1$.

Next, assume that X satisfies the left cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$\begin{aligned}
\beta_2(a_2) * \beta_1(a_2) &= a_2 && \text{(Theorem 3.4.4 } (i)\text{)}\\
&= \beta_2(a_2 * a_1) && \text{(Theorem 3.4.4 } (ii)\text{)}\\
&= \beta_2(a_2) * \beta_2(a_1). && \text{(Theorem 3.4.4 } (iii)\text{)}
\end{aligned}$$

Cancel $\beta_2(a_2)$ from the left so $\beta_1(a_2) = \beta_2(a_1)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.4.11 that $\mid X_1 \cap X_2 \mid = 1$. □

Theorem 3.4.19 *In any anti-internal direct product* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X), X_2 = \beta_2(X)$ *and* $X_1 \cap X_2 = \{d\}$*,* $\beta_1 \mid_{X_1} = \mathbf{1}_{X_1}$ *and* $\beta_2 \mid_{X_2} = \mathbf{1}_{X_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) if and only if* $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall y \in X$*,* $(x_2 * y) * x_1 = x_2 * (y * x_1)$*.*

Proof. Assume that $(x_2 * y) * x_1 = x_2 * (y * x_1)$ for all $x_1 \in X_1, x_2 \in X_2$, and $y \in X$. Choose $x_2 \in X_2$. Let $x_1 \in X_1$. Then

$$\begin{aligned}
\beta_1(x_1) &= \beta_1((x_2 * x_2) * \beta_1(x_1)) && \text{(Theorem 3.4.4 } (ii)\text{)}\\
&= \beta_1(x_2 * (x_2 * \beta_1(x_1))) && \text{(Assumption)}\\
&= \beta_1(x_2) * \beta_1(x_2 * \beta_1(x_1)) && \text{(Theorem 3.4.4 } (iii)\text{)}\\
&= \beta_1(x_2) * \beta_1(x_1) && \text{(Theorem 3.4.4 } (ii)\text{)}\\
&= \beta_1(x_2 * x_1) && \text{(Theorem 3.4.4 } (iii)\text{)}\\
&= x_1. && \text{(Theorem 3.4.4 } (ii)\text{)}
\end{aligned}$$

Hence, $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$. Next, choose $x_1 \in X_1$. Let $x_2 \in X_2$. Then

$$\begin{aligned}
\beta_2(x_2) &= \beta_2(\beta_2(x_2) * (x_1 * x_1)) && \text{(Theorem 3.4.4 } (ii)) \\
&= \beta_2((\beta_2(x_2) * x_1) * x_1) && \text{(Assumption)} \\
&= \beta_2(\beta_2(x_2) * x_1) * \beta_2(x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= \beta_2(x_2) * \beta_2(x_1) && \text{(Theorem 3.4.4 } (ii)) \\
&= \beta_2(x_2 * x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= x_2. && \text{(Theorem 3.4.4 } (ii))
\end{aligned}$$

Hence, $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$.

Conversely, assume that $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$. By Theorem 3.4.15, we have d is both the left identity element for X_1 and the right identity element for X_2. Let $x_1 \in X_1$, $x_2 \in X_2$, and $y \in X$. Then

$$\begin{aligned}
\beta_1((x_2 * y) * x_1) &= \beta_1(x_2 * y) * \beta_1(x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= (\beta_1(x_2) * \beta_1(y)) * \beta_1(x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= \beta_1(y) * \beta_1(x_1) && \text{(Theorems 3.4.11 } (ii) \text{ and 3.4.15)} \\
&= \beta_1(y * x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= \beta_1(x_2) * \beta_1(y * x_1) && \text{(Theorems 3.4.11 } (ii) \text{ and 3.4.15)} \\
&= \beta_1(x_2 * (y * x_1)) && \text{(Theorem 3.4.4 } (iii))
\end{aligned}$$

and

$$\begin{aligned}
\beta_2((x_2 * y) * x_1) &= \beta_2(x_2 * y) * \beta_2(x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= (\beta_2(x_2) * \beta_2(y)) * \beta_2(x_1) && \text{(Theorem 3.4.4 } (iii)) \\
&= \beta_2(x_2) * \beta_2(y) && \text{(Theorems 3.4.11 } (ii) \text{ and 3.4.15)} \\
&= \beta_2(x_2) * (\beta_2(y) * \beta_2(x_1)) && \text{(Theorems 3.4.11 } (ii) \text{ and 3.4.15)}
\end{aligned}$$

$$= \beta_2(x_2) * \beta_2(y * x_1) \qquad \text{(Theorem 3.4.4 } (iii))$$

$$= \beta_2(x_2 * (y * x_1)). \qquad \text{(Theorem 3.4.4 } (iii))$$

Therefore,

$$\begin{aligned}(x_2 * y) * x_1 &= \beta_2((x_2 * y) * x_1) * \beta_1((x_2 * y) * x_1) && \text{(Theorem 3.4.4 } (i))\\ &= \beta_2(x_2 * (y * x_1)) * \beta_1(x_2 * (y * x_1)) \\ &= x_2 * (y * x_1). && \text{(Theorem 3.4.4 } (i))\end{aligned}$$

□

Corollary 3.4.20 *In any anti-internal direct product $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X), X_2 = \beta_2(X)$, and $X_1 \cap X_2 = \{d\}$, if X is a semigroup, then $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$ (i.e., $\beta_1^2 = \beta_1$ and $\beta_2^2 = \beta_2$) with d is the left identity element for X_1 and the right identity element for X_2.*

Proof. Assume that X is a semigroup. By Theorems 3.4.19 and 3.4.15, we have $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$ with d is the left identity element for X_1 and the right identity element for X_2. □

Applying anti-internal direct products to BCC-algebras

Using Theorem 3.4.3, (BCC-2), (BCC-3), and (2.0.1), we get the following theorem.

Theorem 3.4.21 *Let a BCC-algebra $(X; *, 0)$ be the anti-internal direct product of its BCC-subalgebras X_1 and X_2. Then*

(i) $\forall y_1 \in X_1, \forall x_2, y_2 \in X_2, y_2 * y_1 = (x_2 * y_2) * y_1,$

(ii) $\forall x_1, y_1 \in X_1, \forall y_2 \in X_2, x_1 * (y_2 * y_1) = y_2 * (x_1 * y_1),$

(iii) $\forall x_1, y_1 \in X_1, \forall x_2 \in X_2, (x_2 * x_1) * y_1 = x_1 * y_1,$

(iv) $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall y \in X_1 \cap X_2, 0 = (x_2 * y) * (x_1 * y),$

(v) $\forall y_1 \in X_1, \forall y_2 \in X_2, \forall x \in X_1 \cap X_2, y_2 * y_1 = (x * y_2) * (x * y_1).$

Theorem 3.4.22 *Let $(X; *, \beta_1, \beta_2, 0)$ be a BCC-algebra and unary operations β_1 and β_2. The algebra $(X; *)$ is the anti-internal direct product of $\beta_1(X)$ and $\beta_2(X)$ if and only if the algebra $(X; *, \beta_1, \beta_2, 0)$ has the following properties:*

(i) *$\beta_1 = \boldsymbol{1}_X$ is the identity function,*

(ii) *$\beta_2 = \boldsymbol{0}_X$ is the zero function.*

Proof. Assume that $(X; *, \beta_1, \beta_2, 0)$ is a BCC-algebra and unary operations β_1 and β_2. The algebra $(X; *)$ is the anti-internal direct product of $\beta_1(X)$ and $\beta_2(X)$. Then

$$
\begin{aligned}
\beta_2(0) &= \beta_2(0 * 0) && (2.0.1)\\
&= \beta_2(0) * \beta_2(0) && (\text{Theorem 3.4.4 } (iii))\\
&= 0 && (2.0.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_1(0) &= \beta_1(0 * 0) && (2.0.1)\\
&= \beta_1(0) * \beta_1(0) && (\text{Theorem 3.4.4 } (iii))\\
&= 0. && (2.0.1)
\end{aligned}
$$

(ii) Let $x \in X$. Then

$$
\beta_2(x) = \beta_2(\beta_2(x) * \beta_1(0)) \qquad (\text{Theorem 3.4.4 } (ii))
$$

$$= \beta_2(\beta_2(x) * 0)$$
$$= \beta_2(0) \qquad \text{(BCC-3)}$$
$$= 0.$$

Hence, $\beta_2 = \mathbf{0}_X$.

(i) Let $x \in X$. Then

$$x = \beta_2(x) * \beta_1(x) \qquad \text{(Theorem 3.4.4 } (i)\text{)}$$
$$= 0 * \beta_1(x) \qquad ((ii))$$
$$= \beta_1(x). \qquad \text{(BCC-2)}$$

Hence, $\beta_1 = \mathbf{1}_X$.

Conversely, assume that $\beta_1 = \mathbf{1}_X$ and $\beta_2 = \mathbf{0}_X$. Then (i)-(iii) in Theorem 3.4.4 hold. Hence, $(X;*)$ is the anti-internal direct product of $\beta_1(X) = X$ and $\beta_2(X) = \{0\}$. □

By Theorem 3.4.22, we have the following theorem.

Theorem 3.4.23 *Every BCC-algebra* $(X;*,0)$ *is only the anti-internal direct product of* X *and* $\{0\}$.

3.5 Internal direct products of type 2 of BCC-algebras

Internal direct products of type 2 of algebras

In this section, we introduce the concept of the internal direct product of type 2 of algebras by using the binary operation $\boxtimes$ which is defined in Definition 3.2.1.

Definition 3.5.1 An algebra $(X;*)$ is called the *internal direct product* of type 2 of its subalgebras X_1 and X_2 if the mapping

$$\theta : (x_1, x_2) \mapsto x_1 * x_2 \tag{3.5.1}$$

is an isomorphism from the algebra $(X_1 \times X_2; \boxtimes)$ to X.

Then $\theta^{-1} : X \to X_1 \times X_2$ is an isomorphism. Let $\alpha_1 : X \to X_1$ and $\alpha_2 : X \to X_2$ be such that

$$(\forall x \in X)(\theta^{-1}(x) = (\alpha_1(x), \alpha_2(x))). \tag{3.5.2}$$

Lemma 3.5.2 *Let an algebra $(X;*)$ be the internal direct product of type 2 of its subalgebras X_1 and X_2. Then*

(*i*) $\alpha_1(X) = X_1$.

(*ii*) $\alpha_2(X) = X_2$.

We conclude that α_1 and α_2 are surjective.

Proof. (*i*) Clearly, $\alpha_1(X) \subseteq X_1$. Let $x_1 \in X_1$ and choose $x_2 \in X_2$. Since θ^{-1} is surjective, there exists $x \in X$ such that

$$\begin{aligned} (x_1, x_2) &= \theta^{-1}(x) \\ &= (\alpha_1(x), \alpha_2(x)). \end{aligned} \tag{(3.5.2)}$$

Thus $x_1 = \alpha_1(x) \in \alpha_1(X)$, that is, $X_1 \subseteq \alpha_1(X)$. Hence, $\alpha_1(X) = X_1$.

(*ii*) Clearly, $\alpha_2(X) \subseteq X_2$. Let $x_2 \in X_2$ and choose $x_1 \in X_1$. Since θ^{-1} is surjective, there exists $x \in X$ such that

$$(x_1, x_2) = \theta^{-1}(x)$$
$$= (\alpha_1(x), \alpha_2(x)). \qquad ((3.5.2))$$

Thus $x_2 = \alpha_2(x) \in \alpha_2(X)$, that is, $X_2 \subseteq \alpha_2(X)$. Hence, $\alpha_2(X) = X_2$. □

Theorem 3.5.3 *Let an algebra $(X; *)$ be the internal direct product of type 2 of its subalgebras X_1 and X_2. Then $\forall x_1, y_1 \in X_1, \forall x_2, y_2 \in X_2, (x_1 * x_2) * (y_1 * y_2) = (y_1 * x_1) * (y_2 * x_2)$.*

Proof. Let $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Then

$$\begin{aligned}(x_1 * x_2) * (y_1 * y_2) &= \theta(x_1, x_2) * \theta(y_1, y_2) && ((3.5.1))\\ &= \theta((x_1, x_2) \boxtimes (y_1, y_2)) && \text{(Homomorphism)}\\ &= \theta(y_1 * x_1, y_2 * x_2) && \text{(Definition 3.2.1)}\\ &= (y_1 * x_1) * (y_2 * x_2). && ((3.5.1))\end{aligned}$$

□

Theorem 3.5.4 *Let $(X; *, \alpha_1, \alpha_2)$ be an algebra of type $(2, 1, 1)$. Then the algebra $(X; *)$ is the internal direct product of type 2 of $\alpha_1(X)$ and $\alpha_2(X)$ if and only if the algebra $(X; *, \alpha_1, \alpha_2)$ has the following properties:*

(i) $\forall x \in X, \alpha_1(x) * \alpha_2(x) = x$,

(ii) $\forall x_1, x_2 \in X, \alpha_1(x_1) = \alpha_1(\alpha_1(x_1) * \alpha_2(x_2))$ *and* $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1) * \alpha_2(x_2))$, *in particular,* $\forall x_1 \in X_1$, $\forall x_2 \in X_2$, $\alpha_1(x_1 * x_2) = x_1$ *and* $\alpha_2(x_1 * x_2) = x_2$,

(iii) α_1 *and* α_2 *are anti-homomorphisms. Moreover,* $\alpha_1(X_1), \alpha_1(X_2), \alpha_2(X_1)$ *and* $\alpha_2(X_2)$ *are subalgebras of* X.

Proof. Write $\alpha_1(X) = X_1$ and $\alpha_2(X) = X_2$. First, assume that $(X;*)$ is the internal direct product of type 2 of X_1 and X_2.

(i) Let $x \in X$. Then

$$\begin{aligned} x &= \theta(\theta^{-1}(x)) \\ &= \theta(\alpha_1(x), \alpha_2(x)) && ((3.5.2)) \\ &= \alpha_1(x) * \alpha_2(x). && ((3.5.1)) \end{aligned}$$

(ii) Let $x_1, x_2 \in X$. Then there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $\alpha_1(x_1) = y_1$ and $\alpha_2(x_2) = y_2$. Thus

$$\begin{aligned} (\alpha_1(x_1), \alpha_2(x_2)) &= (y_1, y_2) \\ &= \theta^{-1}(\theta(y_1, y_2)) \\ &= \theta^{-1}(y_1 * y_2) && ((3.5.1)) \\ &= (\alpha_1(y_1 * y_2), \alpha_2(y_1 * y_2)) && ((3.5.2)) \\ &= (\alpha_1(\alpha_1(x_1) * \alpha_2(x_2)), \alpha_2(\alpha_1(x_1) * \alpha_2(x_2))). \end{aligned}$$

Hence, $\alpha_1(x_1) = \alpha_1(\alpha_1(x_1) * \alpha_2(x_2))$ and $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1) * \alpha_2(x_2))$.

(iii) Let $x, y \in X$. Then

$$\begin{aligned} (\alpha_1(x * y), \alpha_2(x * y)) &= \theta^{-1}(x * y) && ((3.5.2)) \\ &= \theta^{-1}(x) \boxtimes \theta^{-1}(y) && \text{(Homomorphism)} \\ &= (\alpha_1(x), \alpha_2(x)) \boxtimes (\alpha_1(y), \alpha_2(y)) && ((3.5.2)) \\ &= (\alpha_1(y) * \alpha_1(x), \alpha_2(y) * \alpha_2(x)). && \text{(Definition 3.2.1)} \end{aligned}$$

Hence, $\alpha_1(x * y) = \alpha_1(y) * \alpha_1(x)$ and $\alpha_2(x * y) = \alpha_2(y) * \alpha_2(x)$. Therefore α_1 and α_2 are anti-homomorphisms.

Conversely, assume (i)-(iii) are satisfied. By (iii), we have $\alpha_1(X) = X_1$ and $\alpha_2(X) = X_2$ are subalgebras of X. Define the function $\eta : X \to X_1 \times X_2$ by

$$(\forall x \in X)(\eta(x) = (\alpha_1(x), \alpha_2(x))) \tag{3.5.3}$$

Let $x, y \in X$ be such that $\eta(x) = \eta(y)$. Then

$$\begin{aligned}
\eta(x) = \eta(y) &\Rightarrow \alpha_1(x) = \alpha_1(y) \text{ and } \alpha_2(x) = \alpha_2(y) \\
&\Rightarrow \alpha_1(x) * \alpha_2(x) = \alpha_1(y) * \alpha_2(y) \\
&\Rightarrow x = y. && ((i))
\end{aligned}$$

So, η is injective.

Also, let $(x_1, x_2) \in X_1 \times X_2$. Then $x_1 * x_2 \in X$ such that

$$\begin{aligned}
\eta(x_1 * x_2) &= (\alpha_1(x_1 * x_2), \alpha_2(x_1 * x_2)) && ((3.5.3)) \\
&= (x_1, x_2). && ((ii))
\end{aligned}$$

So, η is surjective.

Also, let $x, y \in X$. Then

$$\begin{aligned}
\eta(x) \boxtimes \eta(y) &= (\alpha_1(x), \alpha_2(x)) \boxtimes (\alpha_1(y), \alpha_2(y)) && ((3.5.3)) \\
&= (\alpha_1(y) * \alpha_1(x), \alpha_2(y) * \alpha_2(x)) && \text{(Definition 3.2.1)} \\
&= (\alpha_1(x * y), \alpha_2(x * y)) && ((iii)) \\
&= \eta(x * y). && ((3.5.3))
\end{aligned}$$

So, η is a homomorphism. Hence, η is an isomorphism and so η^{-1} is an isomorphism.

Finally let $\theta = \eta^{-1}$. Then X is an internal direct product of type 2 of X_1 and X_2. $\square$

Theorem 3.5.5 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X), \alpha_1 |_{X_1}$ *and* $\alpha_2 |_{X_2}$ *are injections.*

Proof. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$\begin{aligned}
(\rho_{\alpha_1(a)} \circ \alpha_1 |_{X_1})(x) &= \rho_{\alpha_1(a)}(\alpha_1 |_{X_1} (x)) \\
&= \rho_{\alpha_1(a)}(\alpha_1(x)) \\
&= \alpha_1(a) * \alpha_1(x) && ((3.3.3)) \\
&= \alpha_1(x * a) && (\text{Theorem 3.5.4 } (iii)) \\
&= x && (\text{Theorem 3.5.4 } (ii)) \\
&= \mathbf{1}_{X_1}(x).
\end{aligned}$$

Thus $\alpha_1 |_{X_1}$ is injective.

Next, choose any $a \in X_1$. Let $x \in X_2$. Then

$$\begin{aligned}
(\lambda_{\alpha_2(a)} \circ \alpha_2 |_{X_2})(x) &= \lambda_{\alpha_2(a)}(\alpha_2 |_{X_2} (x)) \\
&= \lambda_{\alpha_2(a)}(\alpha_2(x)) \\
&= \alpha_2(x) * \alpha_2(a) && ((3.3.4)) \\
&= \alpha_2(a * x) && (\text{Theorem 3.5.4 } (iii)) \\
&= x && (\text{Theorem 3.5.4 } (ii)) \\
&= \mathbf{1}_{X_2}(x).
\end{aligned}$$

Thus $\alpha_2 |_{X_2}$ is injective. $\square$

Corollary 3.5.6 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, for any* $a \in X_2, \rho_{\alpha_1(a)} |_{\alpha_1(X_1)}$ *is a left inverse of* $\alpha_1 |_{X_1}$.

For any $a \in X_1, \lambda_{\alpha_2(a)}|_{\alpha_2(X_2)}$ *is left inverse of* $\alpha_2|_{X_2}$.

Theorem 3.5.7 *In any internal direct product of type 2* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *the following conditions are equivalent.*

(*i*) $\alpha_1(X_1) = X_1$,

(*ii*) *there exists* $d_1 \in X_1$ *such that* $\alpha_1(X_2) = \{d_1\}$ *with* $\alpha_1(d_1) = d_1$,

(*iii*) $\alpha_1(X_1) \cap \alpha_1(X_2) = \{d_1\}$,

(*iv*) $\alpha_1(X_1) \cap \alpha_1(X_2) \neq \emptyset$.

Proof. (*i*)⇒(*ii*) Choose any $x_2 \in X_2$. Then $\alpha_1(\alpha_2(x_2)) \in \alpha_1(X) = X_1 = \alpha_1(X_1)$ from (*i*), so there exists $d_1 \in X_1$ such that $\alpha_1(\alpha_2(x_2)) = \alpha_1(d_1)$. So, let $c_1 \in X_1$. Then

$$\begin{aligned}
\alpha_1(x_2) &= \alpha_1(\alpha_2(c_1 * x_2)) && \text{(Theorem 3.5.4 } (ii))\\
&= \alpha_1(\alpha_2(x_2) * \alpha_2(c_1)) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_1(\alpha_2(c_1)) * \alpha_1(\alpha_2(x_2)) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_1(\alpha_2(c_1)) * \alpha_1(d_1) && (\alpha_1(\alpha_2(x_2)) = \alpha_1(d_1))\\
&= \alpha_1(d_1 * \alpha_2(c_1)) && \text{(Theorem 3.5.4 } (iii))\\
&= d_1. && \text{(Theorem 3.5.4 } (ii))
\end{aligned}$$

So, $\alpha_1(X_2) \subseteq \{d_1\}$. Since $\alpha_1(X_2)$ is nonempty, we have $\alpha_1(X_2) = \{d_1\}$. Also $\alpha_1(d_1) = \alpha_1(\alpha_2(x_2)) \in \alpha_1(X_2) = \{d_1\}$, so $\alpha_1(d_1) = d_1$. Hence, $\alpha_1(X_2) = \{d_1\}$ with $\alpha_1(d_1) = d_1$.

(*ii*)⇒(*iii*) From (*ii*), we have $\alpha_1(X_2) = \{d_1\} = \{\alpha_1(d_1)\} \subseteq \alpha_1(X_1)$. Hence, $\alpha_1(X_1) \cap \alpha_1(X_2) = \{d_1\}$.

(*iii*)⇒(*iv*) Obviously.

$(iv) \Rightarrow (i)$ From (iv), there exist $a_1 \in X_1$ and $a_2 \in X_2$ such that $\alpha_1(a_1) = \alpha_1(a_2)$. Clearly, $\alpha_1(X_1) \subseteq X_1$. So, let $x \in X_1$. Then

$$\begin{aligned} x &= \alpha_1(x * a_2) && \text{(Theorem 3.5.4 } (ii)) \\ &= \alpha_1(a_2) * \alpha_1(x) && \text{(Theorem 3.5.4 } (iii)) \\ &= \alpha_1(a_1) * \alpha_1(x) && (\alpha_1(a_1) = \alpha_1(a_2)) \\ &= \alpha_1(x * a_1) && \text{(Theorem 3.5.4 } (iii)) \\ &\in \alpha_1(X_1). \end{aligned}$$

So, $X_1 \subseteq \alpha_1(X_1)$. Hence, $\alpha_1(X_1) = X_1$. □

Theorem 3.5.8 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$, *the following conditions are equivalent.*

(i) $\alpha_2(X_2) = X_2$,

(ii) *there exists* $d_2 \in X_2$ *such that* $\alpha_2(X_1) = \{d_2\}$ *with* $\alpha_2(d_2) = d_2$,

(iii) $\alpha_2(X_1) \cap \alpha_2(X_2) = \{d_2\}$,

(iv) $\alpha_2(X_1) \cap \alpha_2(X_2) \neq \emptyset$.

Proof. Prove it using the same principle as Theorem 3.5.7, replacing the subscript 1 to 2 and 2 to 1. □

Corollary 3.5.9 *Under the conditions of Theorem 3.5.7* (i)*-*(iv), $\rho_{d_1} |_{X_1}$ *is the inverse automorphism of* $\alpha_1 |_{X_1}$. *Moreover,* $\forall a \in X_2$, $\rho_{d_1} |_{X_1} = \lambda_a |_{X_1}$.

Proof. We shall prove that $\rho_{d_1} |_{X_1}$ is the left inverse automorphism of $\alpha_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$(\rho_{d_1} |_{X_1} \circ \alpha_1 |_{X_1})(x) = \rho_{d_1} |_{X_1} (\alpha_1 |_{X_1} (x))$$

$$= \rho_{d_1}(\alpha_1(x))$$

$$= \rho_{\alpha_1(a)}(\alpha_1(x)) \qquad \text{(Theorem 3.5.7 } (ii))$$

$$= \alpha_1(a) * \alpha_1(x) \qquad ((3.3.3))$$

$$= \alpha_1(x * a) \qquad \text{(Theorem 3.5.4 } (iii))$$

$$= x. \qquad \text{(Theorem 3.5.4 } (ii))$$

Thus $\alpha_1 |_{X_1}$ is injective.

Next, we shall prove that $\lambda_a |_{X_1}$ is the right inverse automorphism of $\alpha_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$(\alpha_1 |_{X_1} \circ \lambda_a |_{X_1})(x) = \alpha_1 |_{X_1} (\lambda_a |_{X_1} (x))$$

$$= \alpha_1 |_{X_1} (\lambda_a(x))$$

$$= \alpha_1(\lambda_a(x))$$

$$= \alpha_1(x * a) \qquad ((3.3.4))$$

$$= x. \qquad \text{(Theorem 3.5.4 } (ii))$$

Thus $\alpha_1 |_{X_1}$ is surjective, so $\alpha_1 |_{X_1}$ is bijective. Hence, $\rho_{d_1} |_{X_1} = \lambda_a |_{X_1}$ is the inverse of $\alpha_1 |_{X_1}$. By Theorem 3.5.4 (iii), we have $\alpha_1 |_{X_1}$ is an anti-automorphism.

□

Corollary 3.5.10 *Under the conditions of Theorem 3.5.7 (i)-(iv), $\lambda_{d_2} |_{X_2}$ is the inverse automorphism of $\alpha_2 |_{X_2}$. Moreover, $\forall a \in X_1$, $\lambda_{d_2} |_{X_2} = \rho_a |_{X_2}$.*

Proof. The proof is in the same way as Corollary 3.5.9. □

Theorem 3.5.11 *In any internal direct product of type 2 $(X; *, \alpha_1, \alpha_2)$ with $X_1 = \alpha_1(X)$ and $X_2 = \alpha_2(X)$, the following conditions are equivalent.*

(i) *$\alpha_1(X_1) = X_1$ and $\alpha_2(X_2) = X_2$,* (*ii*) *there exists* $d \in X$ *such that* $\alpha_1(X_2) = \alpha_2(X_1) = \{d\}$ *with* $\alpha_1(d) = \alpha_2(d) = d$ *and* $d * d = d$,

(*iii*) $X_1 \cap X_2 = \{d\}$,

(*iv*) $X_1 \cap X_2 \neq \emptyset$.

Proof. (*i*)⇒(*ii*) From (*i*), Theorem 3.5.7 gives $\alpha_1(X_2) = \{d_1\}$ with $\alpha_1(d_1) = d_1$ for some $d_1 \in X_1$ and Theorem 3.5.8 gives $\alpha_2(X_1) = \{d_2\}$ with $\alpha_2(d_2) = d_2$ for some $d_2 \in X_2$. Also, $\alpha_1(d_2) \in \alpha_1(X_2) = \{d_1\}$ and $\alpha_2(d_1) \in \alpha_2(X_1) = \{d_2\}$, so $\alpha_1(d_2) = d_1$ and $\alpha_2(d_1) = d_2$. Thus

$$\begin{aligned} d_1 &= \alpha_1(d_1) * \alpha_2(d_1) && \text{(Theorem 3.5.4 } (i)) \\ &= d_1 * d_2 && (\alpha_1(d_1) = d_1, \alpha_2(d_1) = d_2) \\ &= \alpha_1(d_2) * \alpha_2(d_2) && (\alpha_1(d_2) = d_1, \alpha_2(d_2) = d_2) \\ &= d_2. && \text{(Theorem 3.5.4 } (i)) \end{aligned}$$

Choose $d = d_1$. Hence, $\alpha_1(X_2) = \alpha_2(X_1) = \{d\}$ with $\alpha_1(d) = \alpha_2(d) = d$ and $d * d = d$.

(*ii*)⇒(*iii*) From (*ii*), $d \in \alpha_1(X_2) \cap \alpha_2(X_1) \subseteq X_1 \cap X_2$. Thus $\{d\} \subseteq X_1 \cap X_2$. Let $x \in X_1 \cap X_2$. Then $\alpha_1(x) \in \alpha_1(X_1 \cap X_2) \subseteq \alpha_1(X_2) = \{d\}$ and $\alpha_2(x) \in \alpha_2(X_1 \cap X_2) \subseteq \alpha_2(X_1) = \{d\}$. Thus

$$\begin{aligned} x &= \alpha_1(x) * \alpha_2(x) && \text{(Theorem 3.5.4 } (i)) \\ &= d * d && (\alpha_1(x) = d = \alpha_2(x)) \\ &= d. && \text{(Assumption)} \end{aligned}$$

Thus, $X_1 \cap X_2 \subseteq \{d\}$. Hence, $X_1 \cap X_2 = \{d\}$.

(*iii*)⇒(*iv*) Obviously.

$(iv) \Rightarrow (i)$ We see that $\alpha_1(X_1 \cap X_2) \subseteq \alpha_1(X_1) \cap \alpha_1(X_2)$. Thus

$$\begin{aligned} X_1 \cap X_2 \neq \emptyset &\Rightarrow \alpha_1(X_1) \cap \alpha_1(X_2) \neq \emptyset \\ &\Rightarrow \alpha_1(X_1) = X_1 \text{ and } \alpha_2(X_2) = X_2. \end{aligned}$$

(Theorems 3.5.7 (i) and 3.5.8 (i))

Hence, $\alpha_1(X_1) = X_1$ and $\alpha_2(X_2) = X_2$. □

Corollary 3.5.12 *Under the conditions of Theorem 3.5.11 (i)-(iv), $\rho_d \mid_{X_1}$ is the inverse automorphism of $\alpha_1 \mid_{X_1}$ and $\lambda_d \mid_{X_2}$ is the inverse automorphism of $\alpha_2 \mid_{X_2}$. Moreover, $\forall a \in X_2$, $\forall b \in X_1$, $\rho_d \mid_{X_1} = \lambda_a \mid_{X_1}$ and $\lambda_d \mid_{X_2} = \rho_b \mid_{X_2}$.*

Proof. It is a direct result of Corollaries 3.5.9 and 3.5.10. □

Corollary 3.5.13 *In any internal direct product of type 2 $(X; *, \alpha_1, \alpha_2)$ with $X_1 = \alpha_1(X)$ and $X_2 = \alpha_2(X)$, $\mid X_1 \cap X_2 \mid$ can only be 1 or 0.*

Proof. It follows from Theorem 3.5.11 (iii) and (iv). □

Theorem 3.5.14 *In any internal direct product of type 2 $(X; *, \alpha_1, \alpha_2)$ with $X_1 = \alpha_1(X)$ and $X_2 = \alpha_2(X)$, if $X_1 \cap X_2 = \{d\}$, and d commutes with every element of X, then every element of X_1 commutes with every element of X_2.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then

$$\begin{aligned} x_1 * x_2 &= (\alpha_1(x_1) * \alpha_2(x_1)) * (\alpha_1(x_2) * \alpha_2(x_2)) && \text{(Theorem 3.5.4 } (i)) \\ &= (\alpha_1(x_1) * d) * (d * \alpha_2(x_2)) && \text{(Theorem 3.5.11 } (ii)) \\ &= (d * \alpha_2(x_2)) * (\alpha_1(x_1) * d) && \text{(Theorem 3.5.3)} \\ &= (\alpha_1(x_2) * \alpha_2(x_2)) * (\alpha_1(x_1) * \alpha_2(x_1)) && \text{(Theorem 3.5.11 } (ii)) \\ &= x_2 * x_1. && \text{(Theorem 3.5.4 } (i)) \end{aligned}$$

The proof is completed. □

Theorem 3.5.15 *In any internal direct product of type 2* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$*,* $X_2 = \alpha_2(X)$ *and* $X_1 \cap X_2 = \{d\}$*,* $\alpha_1 |_{X_1} = \mathbf{1}_{X_1}$ *and* $\alpha_2 |_{X_2} = \mathbf{1}_{X_2}$ *(i.e.,* $\alpha_1^2 = \alpha_1$ *and* $\alpha_2^2 = \alpha_2$*) if and only if* d *is both a left identity element for* X_1 *and a right identity element for* X_2*.*

Proof. Assume that both $\alpha_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\alpha_2 |_{X_2} = \mathbf{1}_{X_2}$. Then Corollary 3.5.12 gives $\rho_d |_{X_1} = \mathbf{1}_{X_1}$ and $\lambda_d |_{X_2} = \mathbf{1}_{X_2}$. Let $x_1 \in X_1$. Then $d * x_1 = \rho_d(x_1) = \rho_d |_{X_1} (x_1) = \mathbf{1}_{X_1}(x_1) = x_1$, that is, d is the left identity element for X_1. Let $x_2 \in X_2$. Then $x_2 * d = \lambda_d(x_2) = \lambda_d |_{X_2} (x_2) = \mathbf{1}_{X_2}(x_2) = x_2$, that is, d is the right identity element for X_2. Hence, d is both the left identity element for X_1 and the right identity element for X_2.

Conversely, assume that d is both the left identity element for X_1 and the right identity element for X_2. By the assumption, Theorem 3.5.11 (iii) holds. Thus by Theorem 3.5.11, we have the conditions (i)-(iv). It follows from Corollary 3.5.12 that $\rho_d |_{X_1} \circ \alpha_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\alpha_2 |_{X_2} \circ \lambda_d |_{X_2} = \mathbf{1}_{X_2}$. Let $x_1 \in X_1$. Since d is the left identity element for X_1, we have $\mathbf{1}_{X_1}(x_1) = (\rho_d |_{X_1} \circ \alpha_1 |_{X_1})(x_1) = \rho_d |_{X_1} (\alpha_1 |_{X_1} (x_1)) = \rho_d(\alpha_1 |_{X_1} (x_1)) = d * \alpha_1 |_{X_1} (x_1) = \alpha_1 |_{X_1} (x_1)$. Thus $\alpha_1 |_{X_1} = \mathbf{1}_{X_1}$. Next, let $x_2 \in X_2$. Since d is the right identity element for X_2, we have $\mathbf{1}_{X_2}(x_2) = (\alpha_2 |_{X_2} \circ \lambda_d |_{X_2})(x_2) = \alpha_2 |_{X_2} (\lambda_d |_{X_2} (x_2)) = \alpha_2 |_{X_2} (\lambda_d(x_2)) = \alpha_2 |_{X_2} (x_2 * d) = \alpha_2 |_{X_2} (x_2)$. Thus $\alpha_2 |_{X_2} = \mathbf{1}_{X_2}$. □

Theorem 3.5.16 *In any internal direct product of type 2* $(X;*,\alpha_1,\alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, if every element of* X_1 *commutes with every element of* X_2*, and* $\alpha_1^2 = \alpha_1$ *and* $\alpha_2^2 = \alpha_2$*, then* X *has the identity element.*

Proof. Assume that every element of X_1 commutes with every element of X_2, and $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$. By Theorem 3.5.15, we have d is the left identity element for X_1 and d is the right identity element for X_2. Let $x \in X$. Then

$$\begin{aligned} x * d &= (\alpha_1(x) * \alpha_2(x)) * (d * d) && \text{(Theorem 3.5.4 } (i)) \\ &= (d * \alpha_1(x)) * (d * \alpha_2(x)) && \text{(Theorem 3.5.3)} \\ &= \alpha_1(x) * (d * \alpha_2(x)) && (d \text{ is the left identity for } X_1) \\ &= \alpha_1(x) * (\alpha_2(x) * d) && \text{(Assumption)} \\ &= \alpha_1(x) * \alpha_2(x) && (d \text{ is the right identity for } X_2) \\ &= x && \text{(Theorem 3.5.4 } (i)) \end{aligned}$$

and

$$\begin{aligned} d * x &= (d * d) * (\alpha_1(x) * \alpha_2(x)) && \text{(Theorem 3.5.4 } (i)) \\ &= (\alpha_1(x) * d) * (\alpha_2(x) * d) && \text{(Theorem 3.5.3)} \\ &= (\alpha_1(x) * d) * \alpha_2(x) && (d \text{ is the right identity for } X_2) \\ &= (d * \alpha_1(x)) * \alpha_2(x) && \text{(Assumption)} \\ &= \alpha_1(x) * \alpha_2(x) && (d \text{ is the left identity for } X_1) \\ &= x. && \text{(Theorem 3.5.4 } (i)) \end{aligned}$$

Hence, d is the identity element of X. □

Theorem 3.5.17 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, if* X_1 *and* X_2 *are finite, then* $\mid X_1 \cap X_2 \mid = 1$.

Proof. Assume that X_1 and X_2 are finite. By Theorem 3.5.5, we have $\alpha_1 \mid_{X_1}$ and $\alpha_2 \mid_{X_2}$ are injective. By the assumption, we have $\alpha_1(X_1) = \alpha_1 \mid_{X_1} (X_1) = X_1$ and $\alpha_2(X_2) = \alpha_2 \mid_{X_2} (X_2) = X_2$. It follows from Theorem 3.5.11 that $\mid X_1 \cap X_2 \mid = 1$. □

Definition 3.5.18 In an algebra $(X; *)$, the *outside and inside cancellation laws* respectively hold if

$$(\forall a, b, c \in X)(a * b = c * a \Rightarrow b = c), \tag{3.5.4}$$

$$(\forall a, b, c \in X)(b * a = a * c \Rightarrow b = c). \tag{3.5.5}$$

Theorem 3.5.19 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X)$ *and* $X_2 = \alpha_2(X)$*, if* X *satisfies either the outside or inside cancellation law, then* $|X_1 \cap X_2| = 1$.

Proof. Assume that X satisfies the outside cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$\begin{aligned}
\alpha_1(a_1) * \alpha_2(a_1) &= a_1 && \text{(Theorem 3.5.4 } (i)\text{)}\\
&= \alpha_1(a_1 * a_2) && \text{(Theorem 3.5.4 } (ii)\text{)}\\
&= \alpha_1(a_2) * \alpha_1(a_1). && \text{(Theorem 3.5.4 } (iii)\text{)}
\end{aligned}$$

Cancel $\alpha_1(a_1)$ from the outside so $\alpha_2(a_1) = \alpha_1(a_2)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.5.11 that $|X_1 \cap X_2| = 1$.

Next, assume that X satisfies the inside cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$\begin{aligned}
\alpha_1(a_2) * \alpha_2(a_2) &= a_2 && \text{(Theorem 3.5.4 } (i)\text{)}\\
&= \alpha_2(a_1 * a_2) && \text{(Theorem 3.5.4 } (ii)\text{)}\\
&= \alpha_2(a_2) * \alpha_2(a_1). && \text{(Theorem 3.5.4 } (iii)\text{)}
\end{aligned}$$

Cancel $\alpha_2(a_2)$ from the inside so $\alpha_1(a_2) = \alpha_2(a_1)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.5.11 that $|X_1 \cap X_2| = 1$. □

Theorem 3.5.20 *In any internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with*

$X_1 = \alpha_1(X)$, $X_2 = \alpha_2(X)$ and $X_1 \cap X_2 = \{d\}$, $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$ and $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) if and only if for all $x_1 \in X_1, x_2 \in X_2$, and $y \in X$, $(x_1 * y) * x_2 = x_1 * (y * x_2)$.

Proof. Assume that $(x_1 * y) * x_2 = x_1 * (y * x_2)$ for all $x_1 \in X_1, x_2 \in X_2$, and $y \in X$. Choose $x_2 \in X_2$. Let $x_1 \in X_1$. Then

$$
\begin{aligned}
\alpha_1(x_1) &= \alpha_1(\alpha_1(x_1) * (x_2 * x_2)) && \text{(Theorem 3.5.4 }(ii))\\
&= \alpha_1((\alpha_1(x_1) * x_2) * x_2) && \text{(Assumption)}\\
&= \alpha_1(x_2) * \alpha_1(\alpha_1(x_1) * x_2) && \text{(Theorem 3.5.4 }(iii))\\
&= \alpha_1(x_2) * \alpha_1(x_1) && \text{(Theorem 3.5.4 }(ii))\\
&= \alpha_1(x_1 * x_2) && \text{(Theorem 3.5.4 }(iii))\\
&= x_1. && \text{(Theorem 3.5.4 }(ii))
\end{aligned}
$$

Hence, $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$. Next, choose $x_1 \in X_1$. Let $x_2 \in X_2$. Then

$$
\begin{aligned}
\alpha_2(x_2) &= \alpha_2((x_1 * x_1) * \alpha_2(x_2)) && \text{(Theorem 3.5.4 }(ii))\\
&= \alpha_2(x_1 * (x_1 * \alpha_2(x_2))) && \text{(Assumption)}\\
&= \alpha_2(x_1 * \alpha_2(x_2)) * \alpha_2(x_1) && \text{(Theorem 3.5.4 }(iii))\\
&= \alpha_2(x_2) * \alpha_2(x_1) && \text{(Theorem 3.5.4 }(ii))\\
&= \alpha_2(x_1 * x_2) && \text{(Theorem 3.5.4 }(iii))\\
&= x_2. && \text{(Theorem 3.5.4 }(ii))
\end{aligned}
$$

Hence, $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$.

Conversely, assume that $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$ and $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$. By Theorem 3.5.15, we have d is both the left identity element for X_1 and the right identity element for X_2. Let $x_1 \in X_1$, $x_2 \in X_2$, and $y \in X$. Then

$$\begin{aligned}
\alpha_1(x_1 * (y * x_2)) &= \alpha_1(y * x_2) * \alpha_1(x_1) && \text{(Theorem 3.5.4 } (iii))\\
&= (\alpha_1(x_2) * \alpha_1(y)) * \alpha_1(x_1) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_1(y) * \alpha_1(x_1) && \text{(Theorems 3.5.11 } (ii) \text{ and 3.5.15)}\\
&= \alpha_1(x_1 * y) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_1(x_2) * \alpha_1(x_1 * y) && \text{(Theorems 3.5.11 } (ii) \text{ and 3.5.15)}\\
&= \alpha_1((x_1 * y) * x_2) && \text{(Theorem 3.5.4 } (iii))
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2((x_1 * y) * x_2) &= \alpha_2(x_2) * \alpha_2(x_1 * y) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_2(x_2) * (\alpha_2(y) * \alpha_2(x_1)) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_2(x_2) * \alpha_2(y) && \text{(Theorems 3.5.11 } (ii) \text{ and 3.5.15)}\\
&= \alpha_2(y * x_2) && \text{(Theorem 3.5.4 } (iii))\\
&= \alpha_2(y * x_2) * \alpha_2(x_1) && \text{(Theorems 3.5.11 } (ii) \text{ and 3.5.15)}\\
&= \alpha_2(x_1 * (y * x_2)). && \text{(Theorem 3.5.4 } (iii))
\end{aligned}$$

Therefore,

$$\begin{aligned}
(x_1 * y) * x_2 &= \alpha_1((x_1 * y) * x_1) * \alpha_2((x * y) * x_2) && \text{(Theorem 3.5.4 } (i))\\
&= \alpha_1(x_1 * (y * x_2)) * \alpha_2(x_1 * (y * x_2))\\
&= x_1 * (y * x_2). && \text{(Theorem 3.5.4 } (i))
\end{aligned}$$

□

Corollary 3.5.21 *In any anti-internal direct product of type 2* $(X; *, \alpha_1, \alpha_2)$ *with* $X_1 = \alpha_1(X), X_2 = \alpha_2(X)$, *and* $X_1 \cap X_2 = \{d\}$, *if* X *is a semigroup, then*

$\alpha_1 \mid_{X_1} = \boldsymbol{1}_{X_1}$ and $\alpha_2 \mid_{X_2} = \boldsymbol{1}_{X_2}$ (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) with d is the left identity element for X_1 and the right identity element for X_2.

Proof. Assume that X is a semigroup. By Theorems 3.5.20 and 3.5.15, we have $\alpha_1 \mid_{X_1} = \mathbf{1}_{X_1}$ and $\alpha_2 \mid_{X_2} = \mathbf{1}_{X_2}$ with d is the left identity element for X_1 and the right identity element for X_2. □

Applying internal direct products of type 2 of BCC-algebras

We apply the results of the internal direct product of type 2 of algebras to the internal direct product of BCC-algebras of type 2 and get the following results.

Theorem 3.5.22 *Let a BCC-algebra $(X; *, 0)$ be the internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then*

(*i*) $\forall x_1 \in X_1, \forall x_2 \in X_2, x_1 * x_2 = 0,$

(*ii*) $\alpha_2(X) = X_2 = \{0\}.$

Proof. (*i*) Using Theorem 3.5.3, (BCC-2), and (BCC-3), we get the result.

(*ii*) Since $0 \in X_1$, it follows from (*i*) and (BCC-2) that $x = 0 * x = 0$ for all $x \in X_2$. Hence, $X_2 = \{0\}$. □

Theorem 3.5.23 *Let a BCC-algebra $(X; *, 0)$ be the internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then $X = \{0\}$.*

Proof. Assume that a BCC-algebra $(X; *, 0)$ is the internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Let $x \in X$. Then

$$x = \alpha_1(x) * \alpha_2(x) \qquad \text{(Theorem 3.5.4 } (i))$$

$$= \alpha_1(x) * 0 \qquad \text{(Theorem 3.5.22 } (ii))$$

$$= 0. \qquad \text{(BCC-3)}$$

Hence, $X = \{0\}$. □

By Theorem 3.5.23, we have the following theorem.

Theorem 3.5.24 *The only one BCC-algebra that satisfies the internal direct product of type 2 is the zero BCC-algebra* $\{0\}$.

3.6 Anti-internal direct products of type 2 of BCC-algebras

Anti-internal direct products of type 2 of algebras

In this section, we introduce the concept of the anti-internal direct product of type 2 of algebras by using binary operation $\boxtimes$ which is defined in Definition 3.2.1.

Definition 3.6.1 An algebra $(X;*)$ is called the *anti-internal direct product* of type 2 of its subalgebras X_1 and X_2 if the mapping

$$\phi : (x_1, x_2) \mapsto x_2 * x_1 \tag{3.6.1}$$

is an isomorphism from the algebra $(X_1 \times X_2; \boxtimes)$ to X.

Then $\phi^{-1} : X \to X_1 \times X_2$ is an isomorphism. Let $\beta_1 : X \to X_1$ and $\beta_2 : X \to X_2$ be such that

$$(\forall x \in X)(\phi^{-1}(x) = (\beta_1(x), \beta_2(x))). \tag{3.6.2}$$

Lemma 3.6.2 *Let an algebra $(X;*)$ be the anti-internal direct product of type 2 of its subalgebras X_1 and X_2. Then*

(i) $\beta_1(X) = X_1$.

(ii) $\beta_2(X) = X_2$.

We conclude that β_1 and β_2 are surjective.

Proof. (i) Clearly, $\beta_1(X) \subseteq X_1$. Let $x_1 \in X_1$ and choose $x_2 \in X_2$. Since ϕ^{-1} is surjective, there exists $x \in X$ such that

$$\begin{aligned}(x_1, x_2) &= \phi^{-1}(x)\\ &= (\beta_1(x), \beta_2(x)).\end{aligned} \tag{(3.6.2)}$$

Thus $x_1 = \beta_1(x) \in \beta_1(X)$, that is, $X_1 \subseteq \beta_1(X)$. Hence, $\beta_1(X) = X_1$.

(ii) Clearly, $\beta_2(X) \subseteq X_2$. Let $x_2 \in X_2$ and choose $x_1 \in X_1$. Since ϕ^{-1} is surjective, there exists $x \in X$ such that

$$\begin{aligned}(x_1, x_2) &= \phi^{-1}(x)\\ &= (\beta_1(x), \beta_2(x)).\end{aligned} \tag{(3.6.2)}$$

Thus $x_2 = \beta_2(x) \in \beta_2(X)$, that is, $X_2 \subseteq \beta_2(X)$. Hence, $\beta_2(X) = X_2$. □

Theorem 3.6.3 *Let an algebra $(X;*)$ be the anti-internal direct product of type 2 of its subalgebras X_1 and X_2. Then $\forall x_1, y_1 \in X_1, \forall x_2, y_2 \in X_2, (x_2*x_1)*(y_2*y_1) = (y_2 * x_2) * (y_1 * x_1)$.*

Proof. Let $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Then

$$(x_2 * x_1) * (y_2 * y_1) = \phi(x_1, x_2) * \phi(y_1, y_2) \tag{(3.6.1)}$$

$$= \phi((x_1, x_2) \boxtimes (y_1, y_2)) \qquad \text{(Homomorphism)}$$

$$= \phi(y_1 * x_1, y_2 * x_2) \qquad \text{(Definition 3.2.1)}$$

$$= (y_2 * x_2) * (y_1 * x_1). \qquad ((3.6.1))$$

□

Theorem 3.6.4 *Let* $(X; *, \beta_1, \beta_2)$ *be an algebra of type* $(2, 1, 1)$. *Then the algebra* $(X; *)$ *is the anti-internal direct product of type 2 of* $\beta_1(X)$ *and* $\beta_2(X)$ *if and only if the algebra* $(X; *, \beta_1, \beta_2)$ *has the following properties:*

(i) $\forall x \in X, \beta_2(x) * \beta_1(x) = x$,

(ii) $\forall x_1, x_2 \in X, \beta_1(x_1) = \beta_1(\beta_2(x_2) * \beta_1(x_1))$ *and* $\beta_2(x_2) = \beta_2(\beta_2(x_2) * \beta_1(x_1))$, *in particular,* $\forall x_1 \in X_1$, $\forall x_2 \in X_2$, $\beta_1(x_2 * x_1) = x_1$ *and* $\beta_2(x_2 * x_1) = x_2$,

(iii) β_1 *and* β_2 *are anti-homomorphisms. Moreover,* $\beta_1(X_1), \beta_1(X_2), \beta_2(X_1)$ *and* $\beta_2(X_2)$ *are subalgebras of* X.

Proof. Write $\beta_1(X) = X_1$ and $\beta_2(X) = X_2$. First, assume that $(X; *)$ is the anti-internal direct product of type 2 of X_1 and X_2.

(i) Let $x \in X$. Then

$$x = \phi(\phi^{-1}(x))$$

$$= \phi(\beta_1(x), \beta_2(x)) \qquad ((3.6.2))$$

$$= \beta_2(x) * \beta_1(x). \qquad ((3.6.1))$$

(ii) Let $x_1, x_2 \in X$. Then there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $\beta_1(x_1) = y_1$ and $\beta_2(x_2) = y_2$. Thus

$$(\beta_1(x_1), \beta_2(x_2)) = (y_1, y_2)$$

$$
\begin{aligned}
&= \phi^{-1}(\phi(y_1, y_2)) \\
&= \phi^{-1}(y_2 * y_1) && ((3.6.1)) \\
&= (\beta_1(y_2 * y_1), \beta_2(y_2 * y_1)) && ((3.6.2)) \\
&= (\beta_1(\beta_2(x_2) * \beta_1(x_1)), \beta_2(\beta_2(x_2) * \beta_1(x_1))).
\end{aligned}
$$

Hence, $\beta_1(x_1) = \beta_1(\beta_2(x_2) * \beta_1(x_1))$ and $\beta_2(x_2) = \beta_2(\beta_2(x_2) * \beta_1(x_1))$.

(iii) Let $x, y \in X$. Then

$$
\begin{aligned}
(\beta_1(x * y), \beta_2(x * y)) &= \phi^{-1}(x * y) && ((3.6.2)) \\
&= \phi^{-1}(x) \boxtimes \phi^{-1}(y) && \text{(Homomorphism)} \\
&= (\beta_1(x), \beta_2(x)) \boxtimes (\beta_1(y), \beta_2(y)) && ((3.6.2)) \\
&= (\beta_1(y) * \beta_1(x), \beta_2(y) * \beta_2(x)). && \text{(Definition 3.2.1)}
\end{aligned}
$$

Hence, $\beta_1(x * y) = \beta_1(y) * \beta_1(x)$ and $\beta_2(x * y) = \beta_2(y) * \beta_2(x)$. Therefore, β_1 and β_2 are anti-homomorphisms.

Conversely, assume (i)-(iii) are satisfied. By (iii), we have $\beta_1(X) = X_1$ and $\beta_2(X) = X_2$ are subalgebras of X. Define the function $\eta : X \to X_1 \times X_2$ by

$$
(\forall x \in X)(\eta(x) = (\beta_1(x), \beta_2(x))) \qquad (3.6.3)
$$

Let $x, y \in X$ be such that $\eta(x) = \eta(y)$. Then

$$
\begin{aligned}
\eta(x) = \eta(y) &\Rightarrow \beta_1(x) = \beta_1(y) \text{ and } \beta_2(x) = \beta_2(y) \\
&\Rightarrow \beta_2(x) * \beta_1(x) = \beta_2(y) * \beta_1(y) \\
&\Rightarrow x = y. && ((i))
\end{aligned}
$$

So, η is injective.

Also, let $(x_1, x_2) \in X_1 \times X_2$. Then there exists $x_2 * x_1 \in X$ such that

$$\begin{aligned} \eta(x_2 * x_1) &= (\beta_1(x_2 * x_1), \beta_2(x_2 * x_1)) && ((3.6.3)) \\ &= (x_1, x_2). && ((ii)) \end{aligned}$$

So, η is surjective.

Also, let $x, y \in X$. Then

$$\begin{aligned} \eta(x) \boxtimes \eta(y) &= (\beta_1(x), \beta_2(x)) \boxtimes (\beta_1(y), \beta_2(y)) && ((3.6.3)) \\ &= (\beta_1(y) * \beta_1(x), \beta_2(y) * \beta_2(x)) && (\text{Definition 3.2.1}) \\ &= (\beta_1(x * y), \beta_2(x * y)) && ((iii)) \\ &= \eta(x * y). && ((3.6.3)) \end{aligned}$$

So, η is a homomorphism. Hence, η is an isomorphism and so η^{-1} is an isomorphism.

Finally let $\phi = \eta^{-1}$. Then X is an anti-internal direct product of type 2 of X_1 and X_2. □

Theorem 3.6.5 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$, $\beta_1 |_{X_1}$ *and* $\beta_2 |_{X_2}$ *are injective.*

Proof. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$\begin{aligned} (\lambda_{\beta_1(a)} \circ \beta_1 |_{X_1})(x) &= \lambda_{\beta_1(a)}(\beta_1 |_{X_1} (x)) \\ &= \lambda_{\beta_1(a)}(\beta_1(x)) \\ &= \beta_1(x) * \beta_1(a) && ((3.3.4)) \\ &= \beta_1(a * x) && (\text{Theorem 3.6.4 } (iii)) \\ &= x && (\text{Theorem 3.6.4 } (ii)) \end{aligned}$$

$$= \mathbf{1}_{X_1}(x).$$

Thus $\beta_1 \mid_{X_1}$ is injective.

Next, choose any $a \in X_1$. Let $x \in X_2$. Then

$$\begin{aligned}(\rho_{\beta_2(a)} \circ \beta_2 \mid_{X_2})(x) &= \rho_{\beta_2(a)}(\beta_2 \mid_{X_2}(x)) \\ &= \rho_{\beta_2(a)}(\beta_2(x)) \\ &= \beta_2(a) * \beta_2(x) && ((3.3.3)) \\ &= \beta_2(x * a) && (\text{Theorem 3.6.4 } (iii)) \\ &= x && (\text{Theorem 3.6.4 } (ii)) \\ &= \mathbf{1}_{X_2}(x).\end{aligned}$$

Thus $\beta_2 \mid_{X_2}$ is injective. □

Corollary 3.6.6 *In any anti-internal direct product of type 2 $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X)$ and $X_2 = \beta_2(X)$, for any $a \in X_2, \lambda_{\beta_1(a)} \mid_{\beta_1(X_1)}$ is a left inverse of $\beta_1 \mid_{X_1}$. For any $a \in X_1, \rho_{\beta_2(a)} \mid_{\beta_2(X_2)}$ is a left inverse of $\beta_2 \mid_{X_2}$.*

Theorem 3.6.7 *In any anti-internal direct product of type 2 $(X; *, \beta_1, \beta_2)$ with $X_1 = \beta_1(X)$ and $X_2 = \beta_2(X)$, the following conditions are equivalent.*

(*i*) $\beta_1(X_1) = X_1$,

(*ii*) *there exists $d_1 \in X_1$ such that $\beta_1(X_2) = \{d_1\}$ with $\beta_1(d_1) = d_1$,*

(*iii*) $\beta_1(X_1) \cap \beta_1(X_2) = \{d_1\}$,

(*iv*) $\beta_1(X_1) \cap \beta_1(X_2) \neq \emptyset$.

Proof. (*i*)⇒(*ii*) Choose any $x_2 \in X_2$. Then $\beta_1(\beta_2(x_2)) \in \beta_1(X) = X_1 = \beta_1(X_1)$ from (*i*), so there exists $d_1 \in X_1$ such that $\beta_1(\beta_2(x_2)) = \beta_1(d_1)$. So, let $c_1 \in X_1$.

Then

$$\begin{aligned}
\beta_1(x_2) &= \beta_1(\beta_2(x_2 * c_1)) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_1(\beta_2(c_1) * \beta_2(x_2)) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(\beta_2(x_2)) * \beta_1(\beta_2(c_1)) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(d_1) * \beta_1(\beta_2(c_1)) && (\beta_1(\beta_2(x_2)) = \beta_1(d_1))\\
&= \beta_1(\beta_2(c_1) * d_1) && \text{(Theorem 3.6.4 } (iii))\\
&= d_1. && \text{(Theorem 3.6.4 } (ii))
\end{aligned}$$

So, $\beta_1(X_2) \subseteq \{d_1\}$. Since $\beta_1(X_2)$ is nonempty, we have $\beta_1(X_2) = \{d_1\}$. Also $\beta_1(d_1) = \beta_1(\beta_2(x_2)) \in \beta_1(X_2) = \{d_1\}$, so $\beta_1(d_1) = d_1$.

$(ii)\Rightarrow(iii)$ From (ii), we have $\beta_1(X_2) = \{d_1\} = \{\beta_1(d_1)\} \subseteq \beta_1(X_1)$. Hence, $\beta_1(X_1) \cap \beta_1(X_2) = \{d_1\}$.

$(iii)\Rightarrow(iv)$ Obviously.

$(iv)\Rightarrow(i)$ From (iv), there exist $a_1 \in X_1$ and $a_2 \in X_2$ such that $\beta_1(a_1) = \beta_1(a_2)$. Clearly, $\beta_1(X_1) \subseteq X_1$. So, let $x \in X_1$. Then

$$\begin{aligned}
x &= \beta_1(a_2 * x) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_1(x) * \beta_1(a_2) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(x) * \beta_1(a_1) && (\beta_1(a_1) = \beta_1(a_2))\\
&= \beta_1(a_1 * x) && \text{(Theorem 3.6.4 } (iii))\\
&\in \beta_1(X_1).
\end{aligned}$$

So, $X_1 \subseteq \beta_1(X_1)$. Hence, $\beta_1(X_1) = X_1$. □

Theorem 3.6.8 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, the following conditions are equivalent.*

(i) $\beta_2(X_2) = X_2$,

(ii) *there exists* $d_2 \in X_2$ *such that* $\beta_2(X_1) = \{d_2\}$ *with* $\beta_2(d_2) = d_2$,

(iii) $\beta_2(X_1) \cap \beta_2(X_2) = \{d_2\}$,

(iv) $\beta_2(X_1) \cap \beta_2(X_2) \neq \emptyset$.

Proof. Prove it using the same principle as Theorem 3.6.7, replacing the subscript 1 to 2 and 2 to 1. □

Corollary 3.6.9 *Under the conditions of Theorem 3.6.7 (i)-(iv),* $\lambda_{d_1} |_{X_1}$ *is the inverse anti-automorphism of* $\beta_1 |_{X_1}$. *Moreover,* $\forall a \in X_2$, $\lambda_{d_1} |_{X_1} = \rho_a |_{X_1}$.

Proof. We shall prove that $\lambda_{d_1} |_{X_1}$ is the left inverse automorphism of $\beta_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$
\begin{aligned}
(\lambda_{d_1} |_{X_1} \circ \beta_1 |_{X_1})(x) &= \lambda_{d_1} |_{X_1} (\beta_1 |_{X_1} (x)) \\
&= \lambda_{d_1}(\beta_1(x)) \\
&= \lambda_{\beta_1(a)}(\beta_1(x)) && \text{(Theorem 3.6.7 } (ii)) \\
&= \beta_1(x) * \beta_1(a) && ((3.3.4)) \\
&= \beta_1(a * x) && \text{(Theorem 3.6.4 } (iii)) \\
&= x. && \text{(Theorem 3.6.4 } (ii))
\end{aligned}
$$

Thus $\beta_1 |_{X_1}$ is injective.

Next, we shall prove that $\rho_a |_{X_1}$ is the right inverse anti-automorphism of $\beta_1 |_{X_1}$. Choose any $a \in X_2$. Let $x \in X_1$. Then

$$
\begin{aligned}
(\beta_1 |_{X_1} \circ \rho_a |_{X_1})(x) &= \beta_1 |_{X_1} (\rho_a |_{X_1} (x)) \\
&= \beta_1(\rho_a(x))
\end{aligned}
$$

$$= \beta_1(a * x) \qquad ((3.3.3))$$

$$= x. \qquad (\text{Theorem 3.6.4 } (ii))$$

Thus $\beta_1 |_{X_1}$ is surjective, so $\beta_1 |_{X_1}$ is bijective. Hence, $\lambda_{d_1} |_{X_1} = \rho_a |_{X_1}$ is the inverse of $\beta_1 |_{X_1}$. By Theorem 3.6.4 (iii), we have $\beta_1 |_{X_1}$ is an anti-automorphism. □

Corollary 3.6.10 *Under the conditions of Theorem 3.6.8* (i)-(iv), $\rho_{d_2} |_{X_2}$ *is the inverse anti-automorphism of* $\beta_2 |_{X_2}$. *Moreover,* $\forall a \in X_1$, $\rho_{d_2} |_{X_2} = \lambda_a |_{X_2}$.

Proof. The proof is in the same way as Corollary 3.6.9. □

Theorem 3.6.11 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$, *the following conditions are equivalent.*

(i) $\beta_1(X_1) = X_1$ *and* $\beta_2(X_2) = X_2$,

(ii) *there exists* $d \in X$ *such that* $\beta_1(X_2) = \beta_2(X_1) = \{d\}$ *with* $\beta_1(d) = \beta_2(d) = d$ *and* $d * d = d$,

(iii) $X_1 \cap X_2 = \{d\}$,

(iv) $X_1 \cap X_2 \neq \emptyset$.

Proof. $(i)\Rightarrow(ii)$ From (i), Theorem 3.6.7 gives $\beta_1(X_2) = \{d_1\}$ with $\beta_1(d_1) = d_1$ for some $d_1 \in X_1$ and Theorem 3.6.8 gives $\beta_2(X_1) = \{d_2\}$ with $\beta_2(d_2) = d_2$ for some $d_2 \in X_2$. Also, $\beta_1(d_2) \in \beta_1(X_2) = \{d_1\}$ and $\beta_2(d_1) \in \beta_2(X_1) = \{d_2\}$, so $\beta_1(d_2) = d_1$ and $\beta_2(d_1) = d_2$. Thus

$$d_1 = \beta_2(d_1) * \beta_1(d_1) \qquad (\text{Theorem 3.6.4 } (i))$$

$$= d_2 * d_1 \qquad (\beta_2(d_1) = d_2, \beta_1(d_1) = d_1)$$

$$= \beta_2(d_2) * \beta_1(d_2) \qquad (\beta_2(d_2) = d_2, \beta_1(d_2) = d_1)$$

$$= d_2. \qquad \text{(Theorem 3.6.4 } (i))$$

Choose $d = d_1$. Hence, $\beta_1(X_2) = \beta_2(X_1) = \{d\}$ with $\beta_1(d) = \beta_2(d) = d$ and $d * d = d$.

$(ii) \Rightarrow (iii)$ From (ii), $d \in \beta_1(X_2) \cap \beta_2(X_1) \subseteq X_1 \cap X_2$. Thus $\{d\} \subseteq X_1 \cap X_2$. Let $x \in X_1 \cap X_2$. Then $\beta_1(x) \in \beta_1(X_1 \cap X_2) \subseteq \beta_1(X_2) = \{d\}$ and $\beta_2(x) \in \beta_2(X_1 \cap X_2) \subseteq \beta_2(X_1) = \{d\}$. Thus

$$\begin{aligned} x &= \beta_2(x) * \beta_1(x) && \text{(Theorem 3.6.4 } (i)) \\ &= d * d \\ &= d. && \text{(Assumption)} \end{aligned}$$

Thus, $X_1 \cap X_2 \subseteq \{d\}$. Hence, $X_1 \cap X_2 = \{d\}$.

$(iii) \Rightarrow (iv)$ Obviously.

$(iv) \Rightarrow (i)$ We see that $\beta_1(X_1 \cap X_2) \subseteq \beta_1(X_1) \cap \beta_1(X_2)$. Thus

$$\begin{aligned} X_1 \cap X_2 \neq \emptyset &\Rightarrow \beta_1(X_1) \cap \beta_1(X_2) \neq \emptyset \\ &\Rightarrow \beta_1(X_1) = X_1 \text{ and } \beta_2(X_2) = X_2. \\ & \qquad \text{(Theorems 3.6.7 } (i) \text{ and 3.6.8 } (i)) \end{aligned}$$

Hence, $\beta_1(X_1) = X_1$ and $\beta_2(X_2) = X_2$. □

Corollary 3.6.12 *Under the conditions of Theorem 3.6.11 (i)-(iv), $\lambda_d \mid_{X_1}$ is the inverse anti-automorphisms of $\beta_1 \mid_{X_1}$ and $\rho_d \mid_{X_2}$ is the inverse anti-automorphisms of $\beta_2 \mid_{X_2}$. Moreover, $\forall a \in X_2$, $\forall b \in X_1$, $\lambda_d \mid_{X_1} = \rho_a \mid_{X_1}$ and $\rho_d \mid_{X_2} = \lambda_b \mid_{X_1}$.*

Proof. It is a direct result of Corollaries 3.6.9 and 3.6.10. □

Corollary 3.6.13 *In any anti-internal direct product of type 2* $(X;*,\beta_1,\beta_2)$ *with* $X_1=\beta_1(X)$ *and* $X_2=\beta_2(X)$, $|X_1\cap X_2|$ *can only be* 1 *or* 0.

Proof. It follows from Theorem 3.6.11 (*iii*) and (*iv*). □

Theorem 3.6.14 *In any anti-internal direct product* $(X;*,\beta_1,\beta_2)$ *of type 2 with* $X_1=\beta_1(X)$ *and* $X_2=\beta_2(X)$, *if* $X_1\cap X_2=\{d\}$, *and* d *commutes with every element of* X, *then every element of* X_1 *commutes with every element of* X_2.

Proof. Let $x_1\in X_1$ and $x_2\in X_2$. Then

$$\begin{aligned}
x_1*x_2 &= (\beta_2(x_1)*\beta_1(x_1))*(\beta_2(x_2)*\beta_1(x_2)) && \text{(Theorem 3.6.4 } (i))\\
&= (d*\beta_1(x_1))*(\beta_2(x_2)*d) && \text{(Theorem 3.6.11 } (ii))\\
&= (\beta_2(x_2)*d)*(d*\beta_1(x_1)) && \text{(Theorem 3.6.3)}\\
&= (\beta_2(x_2)*\beta_1(x_2))*(\beta_2(x_1)*\beta_1(x_1)) && \text{(Theorem 3.6.11 } (ii))\\
&= x_2*x_1. && \text{(Theorem 3.6.4 } (i))
\end{aligned}$$

The proof is completed. □

Theorem 3.6.15 *In any anti-internal direct product of type 2* $(X;*,\beta_1,\beta_2)$ *with* $X_1=\beta_1(X)$, $X_2=\beta_2(X)$ *and* $X_1\cap X_2=\{d\}$, $\beta_1|_{X_1}=\mathbf{1}_{X_1}$ *and* $\beta_2|_{X_2}=\mathbf{1}_{X_2}$ *(i.e.,* $\beta_1^2=\beta_1$ *and* $\beta_2^2=\beta_2$*) if and only if* d *is both the right identity element for* X_1 *and the left identity element for* X_2.

Proof. Assume that both $\beta_1|_{X_1}=\mathbf{1}_{X_1}$ and $\beta_2|_{X_2}=\mathbf{1}_{X_2}$. Then Corollary 3.6.12 gives $\lambda_d|_{X_1}=\mathbf{1}_{X_1}$ and $\rho_d|_{X_2}=\mathbf{1}_{X_2}$. Let $x_1\in X_1$. Then $x_1*d=\lambda_d(x_1)=\lambda_d|_{X_1}(x_1)=\mathbf{1}_{X_1}(x_1)=x_1$, that is, d is the right identity element for X_1. Let $x_2\in X_2$. Then $d*x_2=\rho_d(x_2)=\rho_d|_{X_2}(x_2)=\mathbf{1}_{X_2}(x_2)=x_2$, that is, d is the left identity element for X_2. Hence, d is both the right identity element for X_1 and the left identity element for X_2.

Conversely, assume that d is both the right identity element for X_1 and the left identity element for X_2. By the assumption, Theorem 3.6.11 (iii) holds. Thus by Theorem 3.6.11, we have the conditions (i)-(iv). It follows from Corollary 3.6.12 that $\lambda_d |_{X_1} \circ \beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} \circ \rho_d |_{X_2} = \mathbf{1}_{X_2}$. Let $x_1 \in X_1$. Since d is the right identity element for X_1, we have $\mathbf{1}_{X_1}(x_1) = (\lambda_d |_{X_1} \circ \beta_1 |_{X_1})(x_1) = \lambda_d |_{X_1} (\beta_1 |_{X_1} (x_1)) = \lambda_d(\beta_1 |_{X_1} (x_1)) = \beta_1 |_{X_1} (x_1) * d = \beta_1 |_{X_1} (x_1)$. Thus $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$. Next, let $x_2 \in X_2$. Since d is the left identity element for X_2, we have $\mathbf{1}_{X_2}(x_2) = (\beta_2 |_{X_2} \circ \rho_d |_{X_2})(x_2) = \beta_2 |_{X_2} (\rho_d |_{X_2} (x_2)) = \beta_2 |_{X_2} (\rho_d(x_2)) = \beta_2 |_{X_2} (d * x_2) = \beta_2 |_{X_2} (x_2)$. Thus $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$. □

Theorem 3.6.16 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$, $X_2 = \beta_2(X)$ *and* $X_1 \cap X_2 = \{d\}$, *if every element of* X_1 *commutes with every element of* X_2, *and* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$, *then* X *has the identity element.*

Proof. Assume that every element of X_1 commutes with every element of X_2, and $\beta_1^2 = \beta_1$ and $\beta_2^2 = \beta_2$. By Theorem 3.6.15, we have d is the right identity element for X_1 and d is the left identity element for X_2. Let $x \in X$. Then

$$\begin{aligned}
x * d &= (\beta_2(x) * \beta_1(x)) * (d * d) && \text{(Theorem 3.6.4 } (i))\\
&= (d * \beta_2(x)) * (d * \beta_1(x)) && \text{(Theorem 3.6.3)}\\
&= \beta_2(x) * (d * \beta_1(x)) && (d \text{ is the left identity for } X_2)\\
&= \beta_2(x) * (\beta_1(x) * d) && \text{(Assumption)}\\
&= \beta_2(x) * \beta_1(x) && (d \text{ is the right identity for } X_1)\\
&= x && \text{(Theorem 3.6.4 } (i))
\end{aligned}$$

and

$$d * x = (d * d) * (\beta_2(x) * \beta_1(x)) \qquad \text{(Theorem 3.6.4 } (i))$$

$$= (\beta_2(x) * d) * (\beta_1(x) * d) \quad \text{(Theorem 3.6.3)}$$
$$= (\beta_2(x) * d) * \beta_1(x) \quad (d \text{ is the right identity for } X_1)$$
$$= (d * \beta_2(x)) * \beta_1(x) \quad \text{(Assumption)}$$
$$= \beta_2(x) * \beta_1(x) \quad (d \text{ is the left identity for } X_2)$$
$$= x. \quad \text{(Theorem 3.6.4 } (i))$$

Hence, d is the identity element of X. □

Theorem 3.6.17 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, if* X_1 *and* X_2 *are finite, then* $| X_1 \cap X_2 |= 1$.

Proof. Assume that X_1 and X_2 are finite. By Theorem 3.6.5, we have $\beta_1 |_{X_1}$ and $\beta_2 |_{X_2}$ are injective. By the assumption, we have $\beta_1(X_1) = \beta_1 |_{X_1} (X_1) = X_1$ and $\beta_2(X_2) = \beta_2 |_{X_2} (X_2) = X_2$. It follows from Theorem 3.6.11 that $| X_1 \cap X_2 |=$ 1. □

Theorem 3.6.18 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X)$ *and* $X_2 = \beta_2(X)$*, if* X *satisfies either the inside or outside cancellation law, then* $| X_1 \cap X_2 |= 1$.

Proof. Assume that X satisfies the inside cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$\beta_2(a_1) * \beta_1(a_1) = a_1 \quad \text{(Theorem 3.6.4 } (i))$$
$$= \beta_1(a_2 * a_1) \quad \text{(Theorem 3.6.4 } (ii))$$
$$= \beta_1(a_1) * \beta_1(a_2). \quad \text{(Theorem 3.6.4 } (iii))$$

Cancel $\beta_1(a_1)$ from the inside so $\beta_2(a_1) = \beta_1(a_2)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.6.11 that $| X_1 \cap X_2 |= 1$.

Next, assume that X satisfies the outside cancellation law. Choose any $a_1 \in X_1$ and $a_2 \in X_2$. Then

$$\begin{aligned}
\beta_2(a_2) * \beta_1(a_2) &= a_2 && \text{(Theorem 3.6.4 } (i))\\
&= \beta_2(a_2 * a_1) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_2(a_1) * \beta_2(a_2). && \text{(Theorem 3.6.4 } (iii))
\end{aligned}$$

Cancel $\beta_2(a_2)$ from the outside so $\beta_1(a_2) = \beta_2(a_1)$, therefore $X_1 \cap X_2 \neq \emptyset$. It follows from Theorem 3.6.11 that $| X_1 \cap X_2 |= 1$. □

Theorem 3.6.19 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X), X_2 = \beta_2(X)$ *and* $X_1 \cap X_2 = \{d\}$, $\beta_1 |_{X_1}= \mathbf{1}_{X_1}$ *and* $\beta_2 |_{X_2}= \mathbf{1}_{X_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) if and only if for all* $x_1 \in X_1, x_2 \in X_2$, *and* $y \in X$, $(x_2 * y) * x_1 = x_2 * (y * x_1)$.

Proof. Assume that $(x_2 * y) * x_1 = x_2 * (y * x_1)$ for all $x_1 \in X_1, x_2 \in X_2$, and $y \in X$. Choose $x_2 \in X_2$. Let $x_1 \in X_1$. Then

$$\begin{aligned}
\beta_1(x_1) &= \beta_1((x_2 * x_2) * \beta_1(x_1)) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_1(x_2 * (x_2 * \beta_1(x_1))) && \text{(Assumption)}\\
&= \beta_1(x_2 * \beta_1(x_1)) * \beta_1(x_2) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(x_1) * \beta_1(x_2) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_1(x_2 * x_1) && \text{(Theorem 3.6.4 } (iii))\\
&= x_1. && \text{(Theorem 3.6.4 } (ii))
\end{aligned}$$

Hence, $\beta_1 |_{X_1}= \mathbf{1}_{X_1}$. Next, choose $x_1 \in X_1$. Let $x_2 \in X_2$. Then

$$\begin{aligned}
\beta_2(x_2) &= \beta_2(\beta_2(x_2) * (x_1 * x_1)) && \text{(Theorem 3.6.4 } (ii))\\
&= \beta_2((\beta_2(x_2) * x_1) * x_1) && \text{(Assumption)}
\end{aligned}$$

$$= \beta_2(x_1) * \beta_2(\beta_2(x_2) * x_1) \qquad \text{(Theorem 3.6.4 } (iii))$$

$$= \beta_2(x_1) * \beta_2(x_2) \qquad \text{(Theorem 3.6.4 } (ii))$$

$$= \beta_2(x_2 * x_1) \qquad \text{(Theorem 3.6.4 } (iii))$$

$$= x_2. \qquad \text{(Theorem 3.6.4 } (ii))$$

Hence, $\beta_2 \mid_{X_2} = \mathbf{1}_{X_2}$.

Conversely, assume that $\beta_1 \mid_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 \mid_{X_2} = \mathbf{1}_{X_2}$. By Theorem 3.6.15, we have d is both the right identity element for X_1 and the left identity element for X_2. Let $x_1 \in X_1$, $x_2 \in X_2$, and $y \in X$. Then

$$\begin{aligned}
\beta_1(x_2 * (y * x_1)) &= \beta_1(y * x_1) * \beta_1(x_2) && \text{(Theorem 3.6.4 } (iii))\\
&= (\beta_1(x_1) * \beta_1(y)) * \beta_1(x_2) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(x_1) * \beta_1(y) && \text{(Theorems 3.6.11 } (ii) \text{ and 3.6.15)}\\
&= \beta_1(x_1) * (\beta_1(y) * \beta_1(x_2)) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_1(x_1) * \beta_1(x_2 * y) && \text{(Theorems 3.6.11 } (ii) \text{ and 3.6.15)}\\
&= \beta_1((x_2 * y) * x_1) && \text{(Theorem 3.6.4 } (iii))
\end{aligned}$$

and

$$\begin{aligned}
\beta_2((x_2 * y) * x_1) &= \beta_2(x_1) * \beta_2(x_2 * y) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_2(x_1) * (\beta_2(y) * \beta_2(x_2)) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_2(y) * \beta_2(x_2) && \text{(Theorems 3.6.11 } (ii) \text{ and 3.6.15)}\\
&= (\beta_2(x_1) * \beta_2(y)) * \beta_2(x_2) && \text{(Theorems 3.6.11 } (ii) \text{ and 3.6.15)}\\
&= \beta_2(y * x_1) * \beta_2(x_2) && \text{(Theorem 3.6.4 } (iii))\\
&= \beta_2(x_2 * (y * x_1)). && \text{(Theorem 3.6.4 } (iii))
\end{aligned}$$

Therefore,

$$\begin{aligned}(x_2 * y) * x_1 &= \beta_2((x_2 * y) * x_1) * \beta_1((x_2 * y) * x_1) && \text{(Theorem 3.6.4 } (i)) \\ &= \beta_2(x_2 * (y * x_1)) * \beta_1(x_2 * (y * x_1)) \\ &= x_2 * (y * x_1). && \text{(Theorem 3.6.4 } (i))\end{aligned}$$

□

Corollary 3.6.20 *In any anti-internal direct product of type 2* $(X; *, \beta_1, \beta_2)$ *with* $X_1 = \beta_1(X), X_2 = \beta_2(X)$*, and* $X_1 \cap X_2 = \{d\}$*, if* X *is a semigroup, then* $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ *and* $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$ *(i.e.,* $\beta_1^2 = \beta_1$ *and* $\beta_2^2 = \beta_2$*) with* d *is the right identity element for* X_1 *and the left identity element for* X_2*.*

Proof. Assume that X is a semigroup. By Theorems 3.6.19 and 3.6.15, we have $\beta_1 |_{X_1} = \mathbf{1}_{X_1}$ and $\beta_2 |_{X_2} = \mathbf{1}_{X_2}$ with d is the right identity element for X_1 and the left identity element for X_2. □

Applying anti-internal direct products of type 2 of BCC-algebras

We apply the results of the anti-internal direct product of type 2 of algebras to the anti-internal direct product of BCC-algebras of type 2 and get the following results.

Theorem 3.6.21 *Let a BCC-algebra* $(X; *, 0)$ *be the anti-internal direct product of type 2 of its BCC-subalgebras* X_1 *and* X_2*. Then*

(*i*) $\forall x_1 \in X_1, \forall x_2 \in X_2, x_2 * x_1 = 0,$

(*ii*) $\beta_1(X) = X_1 = \{0\}.$

Proof. (*i*) Using Theorem 3.6.3, (BCC-2), and (BCC-3), we get the result.

(*ii*) Since $0 \in X_2$, it follows from (*i*) and (BCC-2) that $x = 0 * x = 0$ for all $x \in X_1$. Hence, $X_1 = \{0\}$. □

Theorem 3.6.22 *Let a BCC-algebra* $(X; *, 0)$ *be the anti-internal direct product of type 2 of its BCC-subalgebras* X_1 *and* X_2. *Then* $X = \{0\}$.

Proof. Assume that a BCC-algebra $(X; *, 0)$ is the anti-internal direct product of type 2 of its BCC-subalgebras X_1 and X_2.

Let $x \in X$. Then

$$\begin{aligned} x &= \beta_2(x) * \beta_1(x) && \text{(Theorem 3.6.4 } (i)) \\ &= \beta_2(x) * 0 && \text{(Theorem 3.6.21 } (ii)) \\ &= 0. && \text{(BCC-3)} \end{aligned}$$

Hence, $X = \{0\}$. □

By Theorem 3.6.22, we have the following theorem.

Theorem 3.6.23 *The only one BCC-algebra that satisfies the anti-internal direct product of type 2 is the zero BCC-algebra* $\{0\}$.

CHAPTER V

CONCLUSIONS

In this thesis, we have introduced the concept of the direct product of an infinite family of BCC-algebras, we call the external direct product. We have found the result of the external direct product of special subsets of BCC-algebras. Also, we have introduced the concept of the weak direct product BCC-algebras. Moreover, we have provided the fundamental theorem of (anti-)BCC-homomorphisms in view of the external direct product BCC-algebra. In addition, we have introduced four new concepts of internal direct products of BCC-algebras: the internal direct product, the anti-internal direct product, the internal direct product of type 2, and the anti-internal direct product of type 2. We have studied the properties of four concepts. Finally, we can conclude that for a BCC-algebra $(X; *, 0)$, there is only one form of the internal direct product and only one form of the anti-internal direct product, and finally, there can only be the zero BCC-algebra that satisfies of internal direct product and the anti-internal direct product of type 2 as follows:

1. $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes, (0_i)_{i \in I})$ is a BCC-algebra.

2. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra (resp., meet-commutative BCC-algebra) for all $i \in I$ if and only if $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes, (0_i)_{i \in I})$ is a bounded BCC-algebra (resp., meet-commutative BCC-algebra).

3. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod_{i \in I}^{\mathrm{w}} X_i$ is a BCC-subalgebra of the external direct product BCC-algebra $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes, (0_i)_{i \in I})$.

4. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra (resp., near BCC-filter, BCC-filter, comparative BCC-filter, shift BCC-filter, implicative BCC-filter, BCC-ideal, strong BCC-ideal) of X_i for all $i \in I$ if and only if $\prod_{i \in I} S_i$ is a BCC-subalgebra (resp., near BCC-filter, BCC-filter, comparative BCC-filter, shift BCC-filter, implicative BCC-filter, BCC-ideal, strong BCC-ideal) of the external direct product BCC-algebra $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes, (0_i)_{i \in I})$.

5. Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then

   (i) ψ_i is injective for all $i \in I$ if and only if ψ is injective,

   (ii) ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,

   (iii) ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.

6. Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then

   (i) ψ_i is a (anti-)BCC-homomorphism for all $i \in I$ if and only if ψ is a (anti-)BCC-homomorphism,

   (ii) ψ_i is a (anti-)BCC-monomorphism for all $i \in I$ if and only if ψ is a (anti-)BCC-monomorphism,

   (iii) ψ_i is a (anti-)BCC-epimorphism for all $i \in I$ if and only if ψ is a (anti-)BCC-epimorphism,

   (iv) ψ_i is a (anti-)BCC-isomorphism for all $i \in I$ if and only if ψ is a (anti-)BCC-isomorphism,

   (v) $\ker \psi = \prod_{i \in I} \ker \psi_i$ and $\psi(\prod_{i \in I} X_i) = \prod_{i \in I} \psi_i(X_i)$.

7. $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \boxtimes, (0_i)_{i \in I})$ is a dBCC-algebra.

8. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra (resp., meet-commutative BCC-algebra) for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a bounded dBCC-algebra (resp., join-commutative dBCC-algebra).

9. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is a dBCC-subalgebra of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.

10. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra (resp., near BCC-filter, BCC-filter, comparative BCC-filter, shift BCC-filter, implicative BCC-filter, BCC-ideal, strong BCC-ideal) of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-subalgebra (resp., near dBCC-filter, dBCC-filter, comparative dBCC-filter, shift dBCC-filter, implicative dBCC-filter, dBCC-ideal, strong dBCC-ideal) of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.

11. Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then

    (i) ψ_i is a (anti-)BCC-homomorphism for all $i \in I$ if and only if ψ is a (anti-)dBCC-homomorphism,

    (ii) ψ_i is a (anti-)BCC-monomorphism for all $i \in I$ if and only if ψ is a (anti-)dBCC-monomorphism,

    (iii) ψ_i is a (anti-)BCC-epimorphism for all $i \in I$ if and only if ψ is a (anti-)dBCC-epimorphism,

    (iv) ψ_i is a (anti-)BCC-isomorphism for all $i \in I$ if and only if ψ is a (anti-)dBCC-isomorphism,

    (v) $\ker \psi = \prod_{i\in I} \ker \psi_i$ and $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.

12. Let a BCC-algebra $(X;*,0)$ be the internal direct product of its BCC-subalgebras X_1 and X_2. Then

   (i) $\forall x_1 \in X_1, \forall x_2, y_2 \in X_2, (x_1 * x_2) * y_2 = x_2 * y_2$,

   (ii) $\forall y_1 \in X_1, \forall x_2, y_2 \in X_2, x_2 * (y_1 * y_2) = y_1 * (x_2 * y_2)$,

   (iii) $\forall x_1, y_1 \in X_1, \forall y_2 \in X_2, y_1 * y_2 = (x_1 * y_1) * y_2$,

   (iv) $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall y \in X_1 \cap X_2, 0 = (x_1 * y) * (x_2 * y)$,

   (v) $\forall y_1 \in X_1, \forall y_2 \in X_2, \forall x \in X_1 \cap X_2, y_1 * y_2 = (x * y_1) * (x * y_2)$.

13. Let $(X;*,\alpha_1,\alpha_2,0)$ be a BCC-algebra and unary operations α_1 and α_2. The algebra $(X;*)$ is the internal direct product of $\alpha_1(X)$ and $\alpha_2(X)$ if and only if

   (i) $\alpha_1 = \mathbf{0}_X$ is the zero function,

   (ii) $\alpha_2 = \mathbf{1}_X$ is the identity function.

14. Every BCC-algebra $(X;*,0)$ is only the internal direct product of $\{0\}$ and X.

15. Let a BCC-algebra $(X;*,0)$ be the anti-internal direct product of its BCC-subalgebras X_1 and X_2. Then

   (i) $\forall y_1 \in X_1, \forall x_2, y_2 \in X_2, y_2 * y_1 = (x_2 * y_2) * y_1$,

   (ii) $\forall x_1, y_1 \in X_1, \forall y_2 \in X_2, x_1 * (y_2 * y_1) = y_2 * (x_1 * y_1)$,

   (iii) $\forall x_1, y_1 \in X_1, \forall x_2 \in X_2, (x_2 * x_1) * y_1 = x_1 * y_1$,

   (iv) $\forall x_1 \in X_1, \forall x_2 \in X_2, \forall y \in X_1 \cap X_2, 0 = (x_2 * y) * (x_1 * y)$,

   (v) $\forall y_1 \in X_1, \forall y_2 \in X_2, \forall x \in X_1 \cap X_2, y_2 * y_1 = (x * y_2) * (x * y_1)$.

16. Let $(X;*,\beta_1,\beta_2,0)$ be a BCC-algebra and unary operations β_1 and β_2. The algebra $(X;*)$ is the anti-internal direct product of $\beta_1(X)$ and $\beta_2(X)$ if and only if the algebra $(X;*,\beta_1,\beta_2,0)$ has the following properties:

(i) $\beta_1 = \mathbf{1}_X$ is the identity function,

(ii) $\beta_2 = \mathbf{0}_X$ is the zero function.

17. Every BCC-algebra $(X; *, 0)$ is only the anti-internal direct product of X and $\{0\}$.

18. Let a BCC-algebra $(X; *, 0)$ be the internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then

    (i) $\forall x_1 \in X_1, \forall x_2 \in X_2, x_1 * x_2 = 0$,

    (ii) $\alpha_2(X) = X_2 = \{0\}$.

19. Let a BCC-algebra $(X; *, 0)$ be the internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then $X = \{0\}$.

20. The only one BCC-algebra that satisfies the internal direct product of type 2 is the zero BCC-algebra $\{0\}$.

21. Let a BCC-algebra $(X; *, 0)$ be the anti-internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then

    (i) $\forall x_1 \in X_1, \forall x_2 \in X_2, x_2 * x_1 = 0$,

    (ii) $\beta_1(X) = X_1 = \{0\}$.

22. Let a BCC-algebra $(X; *, 0)$ be the anti-internal direct product of type 2 of its BCC-subalgebras X_1 and X_2. Then $X = \{0\}$.

23. The only one BCC-algebra that satisfies the anti-internal direct product of type 2 is the zero BCC-algebra $\{0\}$.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Abebe, G. A. (2018). On the theory of *BRK*-algebras. [**Doctoral dissertation**]. Department of Mathematics, Addis Ababa University.

[2] Ansari, M. A., Haidar, A., and Koam, A. N. A. (2018). On a graph associated to UP-algebras. **Math. Comput. Appl.**, 23(4), 61.

[3] Ansari, M. A., Koam, A. N. A., and Haider, A. (2019). Rough set theory applied to UP-algebras. **Ital. J. Pure Appl. Math.**, 42, 388–402.

[4] Chanmanee, C., Chinram, R., Prasertpong, R., Julatha, P., and Iampan, A. (2022). Direct product of infinite family of B-algebras, **Eur. J. Pure Appl. Math.**, 15(3), 999–1014.

[5] Chanmanee, C., Prasertpong, R., Julatha, P., Lekkoksung, N. and Iampan, A. On external direct products of IUP-algebras, accepted.

[6] Chanmanee, C., Prasertpong, R., Julatha, P., Kalyani, U. V., Eswarlal, T. and Iampan, A. (2023). A note on external direct products of BP-algebras. **Math. Stat.**, 11(1), 206–212.

[7] Chinram, R. and Iampan, A. (2021). Codewords generated by UP-valued functions. **AIMS Math.**, 6(5), 4771–4785.

[8] Endam, J. C. and Teves, R. C. (2016). Direct product of BF-algebras. **Int. J. Algebra**, 10(3), 125–132.

[9] Guntasow, T., Sajak, S., Jomkham, A., and Iampan, A. (2017). Fuzzy translations of a fuzzy set in UP-algebras. **J. Indones. Math. Soc.**, 23(2), 1–19.

[10] Huang, Y. S. (2006). BCI-algebra, Science Press, Beijing.

[11] Iampan, A. (2017). A new branch of the logical algebra: UP-algebras. **J. Algebra Relat. Top.**, 5(1), 35–54.

[12] Iampan, A. (2018). Introducing fully UP-semigroups. **Discuss. Math., Gen. Algebra Appl.**, 38(2), 297–306.

[13] Iampan, A. (2021). Multipliers and near UP-filters of UP-algebras. **J. Discrete Math. Sci. Cryptography**, 24(3), 667–680.

[14] Iampan, A., Satirad, A., and Songsaeng, M. (2020). A note on UP-hyperalgebras. **J. Algebr. Hyperstruct. Log. Algebr.**, 1(2), 77–95.

[15] Iampan, A., Songsaeng, M., and Muhiuddin, G. (2020). Fuzzy duplex UP-algebras. **Eur. J. Pure Appl. Math.**, 13(3), 459–471.

[16] Imai, Y. and Iséki, K. (1966). On axiom systems of propositional calculi, XIV. **Proc. Japan Acad.**, 42(1), 19–22.

[17] Iséki, K. (1966). An algebra related with a propositional calculus. **Proc. Japan Acad.**, 42(1), 26–29.

[18] Jakubík, J. and Csontóová, M. (2000). Cancellation rule for internal direct product decompositions of a connected partially ordered set. **Math. Bohem.**, 125(1), 115–122.

[19] Jun, Y. B., Brundha, B., Rajesh, N., Bandaru, R. K. (2022). (3, 2)-Fuzzy UP-subalgebras and (3, 2)-fuzzy UP-filters. **J. Mahani Math. Res. Cent.**, 11, 1–14.

[20] Jun, Y. B. and Iampan, A. (2019). Comparative and allied UP-filters. **Lobachevskii J. Math.**, 40(1), 60–66.

[21] Jun, Y. B. and Iampan, A. (2019). Implicative UP-filters. **Afr. Mat.**, 30(7-8), 1093–1101.

[22] Jun, Y. B. and Iampan, A. (2019). Shift UP-filters and decompositions of UP-filters in UP-algebras. **Missouri J. Math. Sci.**, 31(1), 36–45.

[23] Kamuti, I. N. (2012). Cycle index of internal direct product groups. **Int. Math. Forum**, 7(30), 1491–1494.

[24] Karaçal, F. and Khadjiev, D. (2015). Internal direct product of integral $\vee$-distributive binary aggregation functions and the classification of all integral $\vee$-distributive binary aggregation functions of length 3. **Inf. Sci.**, 298, 22–35.

[25] Kavitha, J. and Gowri, R. (2018). Direct product of GK algebra. **Indian J. Technol.**, 14(35), 2802–2805.

[26] Koam, A. N. A., Haider, A., and Ansari, M. A. (2022). Generalized cut functions and n-ary block codes on UP-algebras. **J. Math.**, 2022, Article ID 4789155, 5 pages.

[27] Komori, Y. (1984). The class of BCC-algebras is not a variety. **Math. Japon.**, 29, 391–394.

[28] Lingcong, J. A. V. (2017). On the direct product of B-algebras. [**Doctoral dissertation**]. Negros Oriental State University.

[29] Lingcong, J. A. V. and Endam, J. C. (2016). Direct product of B-algebras. **Int. J. Algebra**, 10(1), 33–40.

[30] Lingcong, J. A. V. and Endam, J. C. (2016). Mappings of the direct product of B-algebras. **Int. J. Algebra**, 10(3), 133–140.

[31] Makamba, B. B. (1992). Direct products and isomorphism of fuzzy subgroups. **Inf. Sci.**, 65, 33–43.

[32] Nama, N. M. (2017). On fuzzy internal direct product. **IBN Al-Haitham Pure Appl. Sci.**, 23(3), 238–245.

[33] Pledger, K. E. (1999). Internal direct products of groupoids. **J. Algebra**, 217, 599–627.

[34] Satirad, A., Chinram, R., and Iampan, A. (2020). Four notions of extension of KU/UP-algebras. **Missouri J. Math. Sci.**, 32(2), 138–157.

[35] Satirad, A. and Iampan, A. (2019). Topological UP-algebras. **Discuss. Math., Gen. Algebra Appl.**, 39(2), 231–250.

[36] Satirad, A., Mosrijai, P., and Iampan, A. (2019). Formulas for finding UP-algebras. **Int. J. Math. Comput. Sci.**, 14(2), 403–409.

[37] Satirad, A., Mosrijai, P., and Iampan, A. (2019). Generalized power UP-algebras. **Int. J. Math. Comput. Sci.**, 14(1), 17–25.

[38] Sawika, K., Intasan, R., Kaewwasri, A., and Iampan, A. (2016). Derivations of UP-algebras. **Korean J. Math.**, 24(3), 345–367.

[39] Senapati, T., Jun, Y. B., and Shum, K. P. (2018). Cubic set structure applied in UP-algebras. **Discrete Math. Algorithms Appl.**, 10(4), Article ID 1850049.

[40] Senapati, T., Muhiuddin, G., and Shum, K. P. (2017). Representation of UP-algebras in interval-valued intuitionistic fuzzy environment. **Ital. J. Pure Appl. Math.**, 38, 497–517.

[41] Setiani, A., Gemawati, S., and Deswita, L. (2020). Direct product of BP-algebra. **Int. J. Math. Trends Technol.**, 66(2), 63–66.

[42] Shalla, M. M. and Olgun, N. (2019). Direct and semi-direct product of neutrosophic extended triplet group. **Neutrosophic Sets Syst.**, 29, 197–214.

[43] Somjanta, J., Thuekaew, N., Kumpeangkeaw, P., and Iampan, A. (2016). Fuzzy sets in UP-algebras. **Ann. Fuzzy Math. Inform.**, 12(6), 739–756.

[44] Judson, T. W. and Robert, A. B. (2022). Abstract algebra: theory and applications. **Orthogonal Publishing L3c**, Ann Arbor, Michigan.

[45] Widianto, S., Gemawati, S., and Kartini (2019). Direct product in BG-algebras. **Int. J. Algebra**, 13(5), 239–247.

BIOGRAPHY

BIOGRAPHY

Name Surname	Chatsuda Chanmanee
Date of Birth	November 13, 1997
Place of Birth	Kamphaeng Phet Province, Thailand
Address	202 Moo 1, Saen To Subdistrict, Khanu Woralaksaburi District, Kamphaeng Phet Province, Thailand 62130

Education Background

2023	M.Sc. (Mathematics), University of Phayao, Phayao, Thailand
2021	B.Ed. (Mathematics), Kamphaeng Phet Rajabhat University, Kamphaeng Phet, Thailand
2020	B.Sc. (Mathematics), Kamphaeng Phet Rajabhat University, Kamphaeng Phet, Thailand

Publications

Articles

1. **Chanmanee, C.**, Chinram, R., Phrommarat, N., Julatha, P., Lekkoksung, N., and Iampan, A. (2023). External and weak direct products of UP (BCC)-algebras. **J. Math. Comput. Sci., JMCS**, 29(1), 90–105.

2. **Chanmanee, C.**, Chinram, R., Prasertpong, R., Julatha, P., and Iampan, A. (2023). External direct products on dual UP (BCC)-algebras. **J. Math. Comput. Sci., JMCS**, 29(2), 175–191.

3. **Chanmanee, C.**, Chinram, R., Prasertpong, R., Julatha, P., and Iampan, A. (2023). Two new concepts of internal direct prod-

ucts of UP (BCC)-algebras. **IAENG Int. J. Comput. Sci.**, 50(1), 306–314.

Others

1. **Chanmanee, C.**, Chinram, R., Prasertpong, R., Julatha, P., and Iampan, A. (2022). (Revised). Type 2 of internal direct products of UP (BCC)-algebras.