FIXED POINT ALGORITHMS FOR NONSELF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS AND G-NONEXPANSIVE MAPPINGS

KRITSADAPHIWAT WONGYAI

A Thesis Submitted to University of Phayao

in Partial Fulfillment of the Requirements

for the Master of Science Degree in Mathematics

May 2020

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Submitted by Kritsadaphiwat Wongyai

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University of Phayao

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May 2020

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เรื่อง: ขั้นตอนวิธีจุดตรึงสำหรับการส่งนอกตัววางนัยทั่วไปไม่ขยายคล้ายแบบเชิงเส้นกำกับและการส่งไม่ขยายแบบ G
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คำสำคัญ: การส่งนอกตัววางนัยทั่วไปไม่ขยายคล้ายแบบเชิงเส้นกำกับ, การส่งไม่ขยายแบบ G, การลู่เข้าแบบเข้ม, ความต่อเนื่องแบบบริบูรณ์, จุดตรึง

บทคัดย่อ

ทฤษฎีจุดตรึงมีการศึกษากันอย่างกว้างขวาง เนื่องจากเป็นเครื่องมือที่มีประโยชน์ในการแก้ปัญหาต่างๆ ในหลากหลายสาขา เช่น วิศวกรรม เศรษฐศาสตร์ เคมี ทฤษฎีเกม และทฤษฎีกราฟ เป็นต้น อย่างไรก็ตามเมื่อการศึกษาเรื่องการมีจริงของจุดตรึงสำหรับบางการส่ง พบว่า การหาค่าของจุดตรึงที่มีอยู่นั้น ไม่ใช่เรื่องง่าย นั่นคือเหตุผลว่า ทำไมจึงใช้กระบวนการทำซ้ำสำหรับการคำนวณหาจุดตรึง กระบวนการทำซ้ำหลายแบบได้รับการพัฒนาขึ้นแต่ก็ไม่ครอบคลุมการส่งต่างๆ ทั้งหมด ที่ทราบกันเป็นอย่างดี ก็คือ ทฤษฎีการหดตัวของบานาคใช้กระบวนการทำซ้ำของปีการ์สำหรับการประมาณค่าของจุดตรึง นอกจากนี้ยังมีกระบวนการทำซ้ำที่รู้จักกันเป็นอย่างดี ได้แก่ กระบวนการทำซ้ำของ มานน์ อิชิคาวา อัลกาวอร์ นูร์ และอื่น ๆ

วัตถุประสงค์แรกของวิทยานิพนธ์นี้ ได้แนะนำและศึกษาระเบียบวิธีการทำซ้ำสองขั้นตอนแบบใหม่ ซึ่งเรียกว่า ระเบียบวิธีการทำซ้ำแบบอิชิคาวาด้วยการรบกวนสำหรับการส่งนอกตัววางนัยทั่วไปไม่ขยายคล้ายแบบเชิงเส้นกำกับในปริภูมิบานาค โดยให้เงื่อนไขที่เพียงพอสำหรับการลู่เข้าของกระบวนการทำซ้ำที่แนะนำขึ้นไปยังจุดตรึงร่วมของการส่ง ภายใต้เงื่อนไขที่กำหนดขึ้นในปริภูมิบานาคนูนเอกรูปค่าจริง ยิ่งไปกว่านั้น ได้แสดงการพิสูจน์การลู่เข้าอย่างเข้มของระเบียบวิธีการทำซ้ำแบบใหม่ด้วยการรบกวน ไปยังจุดตรึงร่วมของสองการส่งนอกตัววางนัยทั่วไปไม่ขยายคล้ายแบบเชิงเส้นกำกับบนเซตย่อยนูนปิด ที่ไม่เป็นเซตว่างของปริภูมิบานาคค่าจริง

วัตถุประสงค์ที่สองได้แนะนำและศึกษาการวิเคราะห์การลู่เข้าของกระบวนการทำซ้ำสองขั้นตอนแบบใหม่เมื่อประยุกต์ไปยังการส่งชนิดการส่งไม่ขยายแบบ G โดยให้ทฤษฎีบทการลู่เข้าอย่างอ่อนและอย่างเข้มสำหรับระเบียบวิธีการทำซ้ำสองขั้นตอนแบบใหม่ในปริภูมิบานาคนูนเอกรูปด้วยกราฟระบุทิศ ยิ่งไปกว่านั้น ได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบอ่อนโดยไม่ใช้เงื่อนไขของโอเปียล และแสดงการทดลองเชิงตัวเลขเพื่อยืนยันผลลัพธ์ที่ได้และเปรียบเทียบอัตราการลู่เข้าของวิธีการทำซ้ำที่แนะนำขึ้น กับวิธีการทำซ้ำแบบอิชิคาวาและวิธีการทำซ้ำปรับปรุงแบบ S

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ABSTRACT

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry, game theory and graph theory etc. However, once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them. By time, many iterative processes have been developed and it is impossible to cover them all. The well-known Banach contraction theorem use Picard iterative process for approximation of fixed point. Some of the well-known iterative processes are those of Mann, Ishikawa, Agarwal, Noor, and so on.

The first purpose of this dissertation is to introduce and study a new type of two-step iterative scheme which is called the projection type Ishikawa iteration with perturbations for two nonself generalized asymptotically quasi-nonexpansive mappings in Banach spaces. A sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is also established in a real uniformly convex Banach space. Furthermore, the strong convergence of a new iterative scheme with perturbations to a common fixed point of two nonself generalized asymptotically quasi-nonexpansive mappings on a nonempty closed convex subset of a real Banach space is proved.

The second purpose is to introduce and study convergence analysis of a new two-step iteration process when applied to class of G-nonexpansive mappings. Weak and strong convergence theorems are established for the new two-step iterative scheme in a uniformly convex Banach space with a directed graph. Moreover, weak convergence theorem without making use of the Opial's condition is proved. We also show the numerical experiment for supporting our main results and comparing rate of convergence of the proposed method with the Ishikawa iteration and the modified S-iteration.

The results obtained in this dissertation extend and generalize some results in the literature.

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CHAPTER I

INTRODUCTION

The presence or absence of a fixed point is an intrinsic property of a map. However, many necessary or sufficient conditions for the existence of such points involve a mixture of algebraic, order theoretic, or topological properties of the mapping or its domain.

The origins of the theory, which date to the latter part of the nineteenth century, test in the use of successive approximations to establish the existence and uniqueness of solutions, particularly to differential equations. This method is associated with the names of such celebrated mathematician as Cauchy, Liouville, Lipschitz, Peano, Fredholm and, expecially, Picard. However, it is the Polish mathematician Stefan Banach who is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. Around 1922, Banach recognized the fundamental role of metric completeness; a property shared by all of the spaces commonly exploited in analysis. For many years, activity in metric fixed point theory was limited to minor extensions of Banach's contraction mapping principal and its manifold applications. The theory gained new impetus largely as a result of pioneering work of Felix Browder in the mid-nineteen sixties and the development of nonlinear functional analysis as an active and vital branch of mathematics. Pivotal in this development were the 1965 existence theorems of Browder, Gohde, and Kirk and the eary metric results of Edelstein. By the end of the decade, a rich fixed point theory for nonexpansive mapping was clearly emerging and it was equally clear that such mappings played a fundamental role in many aspects of nonlinear functional analysis with links to variational inequalities and the theory of monotone and accretive operators.

Nonexpansive mappings respresent the limiting case in the theory of contractions, where the Lipschitz constant is allows to become one, and it was clear from the outset that the study of such mappings required techniques going far beyond purely metric arguments. The theory of nonexpansive mappings has involved an intertwining of geometrical and topological arguments. The original theorems of Browder and Göhde exploited special convexity properties of the norm in certain Banach spaces, while Kirk identified the underlying property of normal structure and the role played by weak compactness. The early phases of the development centred around the identification of spaces whose bounded convex sets possessed normal structure, and it was soon discovered that certain weakenings and variants of normal structure also sufficed. By the mid-nineteen seventies it was apparent that normal structure was a substantially stronger condition than necessary. And, armed with the then newly descovered Goebel Karlovitz lemma the quest turned toward classifying those Banach spaces in which all nonexpansive self-mappings of a nonempty weakly compact convex subset have a fixed point. This has yielded many elegant results and led to numerous discoveries in Banach space geometry, although the question itself remains open. Asymptotic regularity of the averaged map was an important contribution of the late seventies, that has been exploited in many subsequent arguments.

As we know, iteration methods are numerical procedures which compute a sequence of gradually accurate iterates to approximate the solution of a class of problems. Such methods are useful tools of applied mathematics for solving real life problems ranging from economics and finance or biology to transportation, network analysis or optimization. When we design iteration methods, we have to study their qualitative properties such as: convergence, stability, error propagation, stopping criteria. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods; please, (see [[11], [12], [14], [15]]).

Fixed-point iteration process for nonexpansive self-mappings including Mann and Ishikawa iteration processes have been studied extensively by various authors (see [[16], [17], [18]]). We know that Mann and Ishikawa iteration processes are defined as:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1.1}$$

and

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

respectively. Obviously the iterative schemes (1.1) and (1.2) deals with one self-mapping only. In 1986, Das and Debata [17] introduced and studied the case of two mapping in iteration processes. This success can be rich source of inspriation for many authors, see for example, Takahashi and Tamura [46] and Khan and Takahashi [24]. For approximating the common fixed points, the two mappings case has its own importance as it has a direct link with the minimization problem, see for example Takahashi [45].

Being an important generalization of the class of nonexpansive self-mappings, in 1972, Goebel and Kirk [21] introduced the class of asymptotically nonexpansive self-mappings, who proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on C, then T has a fixed point.

In 1991, Schu [36] introduced the following modified Mann iteration process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \tag{1.3}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach spaces (see [30, 36, 34, 47]).

Let C be a nonempty closed convex subset of real normed linear space X. A self-mapping $T : C \to C$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. A self-mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.4}$$

for all $x, y \in C$ and $n \geq 1$. A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$.

It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly $L-$Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

Definition 1.0.1 (see [38]). A self-mapping $T : C \to C$ is called generalized asymptotically nonexpansive if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1$, $k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + \delta_n \tag{1.6}$$

for all $x, y \in C$ and $n \geq 1$. $T : C \to C$ is said to be generalized asymptotically quasi-nonexpansive if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1$, $k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n p\| \leq k_n \|x - p\| + \delta_n \tag{1.7}$$

for all $x \in C$, $p \in F(T)$ ($F(T)$ denote the set of fixed points of T) and $n \geq 1$.

It is clear from the definition that a generalized asymptotically quasi-nonexpansive mapping is to unify various definitions of classes of mappings associated with the class of generalized asymptotically nonexpansive mapping, asymptotically nonexpansive type, asymptotically nonexpansive mappings, and nonexpansive mappings. However, the converse of each of above statement may be not true. The example shows that a generalized asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping; see [38].

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations, for example, asymptotically nonexpansive mappings, etc., have been studied by a number of authors (see, e.g., [13–17]) and references cited therein.

In most of these papers, the well known Mann iteration process (1.1) (see [27]) has been studied and the operator T has been assumed to map C into itself. The convexity of C then ensures that the sequence $\{x_n\}$ generated by (1.1) is well defined. If, however, C is a proper subset of the real Banach space X and T maps C into X (as is the case in many applications), then the sequence given by (1.1) may not be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P : X \to C$ in the recursion formula (1.1) as follows: $x_1 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n PTx_n, \quad n \geq 1. \tag{1.8}$$

For nonself nonexpansive mappings, some authors (see [19–23]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach space.

The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume, Ofoedu and Zegeye [12] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.0.2 (see [12])**.** Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.9}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\| \tag{1.10}$$

for all $x, y \in C$ and $n \geq 1$.

In [12], they studied the following iterative sequence: $x_1 \in C$,

$$x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \tag{1.11}$$

to approximate some fixed point of T under suitable conditions.

If T is a self-mapping, then P becomes the identity mapping so that (1.9) and (1.10) reduce to (1.4) and (1.5), respectively. (1.11) reduces to (1.3).

In 2006, Wang [52] generalizes the iteration process (1.11) as follows: $x_1 \in C$,

$$\begin{aligned} y_n &= P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \quad n \geq 1, \end{aligned} \tag{1.12}$$

where $T_1, T_2 : C \to X$ are nonself asymptotically nonexpansive mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$. He proved strong convergence of the sequence $\{x_n\}$ defined by (1.12) to a common fixed point of T_1 and T_2 under proper conditions. Meanwhile, the results of [52] generalized the results of [12].

The nonself generalized asymptotically nonexpansive and nonself generalized asymptotically quasi-nonexpansive mappings are defined as follows:

Definition 1.0.3 (see [19])**.** Let C be a nonempty subset of a real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself-mapping $T : C \to X$ is called generalized asymptotically nonexpansive if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1$, $k_n \to 1$ and $\delta_n \to 0$

as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x-y\| + \delta_n \tag{1.13}$$

for all $x, y \in C$ and $n \geq 1$. $T : C \to X$ is said to be generalized asymptotically quasi-nonexpansive if there exists nonnegative real sequences $\{k_n\}$ and $\{\delta_n\}$ with $k_n > 1$, $k_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}p\| \leq k_n\|x-p\| + \delta_n \tag{1.14}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

If T is a self-mapping, then P becomes the identity mappings so that (1.13) and (1.14) reduces to (1.6) and (1.7), respectively.

In 2008, Deng and Liu [19] studied the following iterative sequence which can be viewed as an extension for iterative schemes of Wang [52]: $x_i \in C$ ($i = 0, 1, 2, \ldots, q$ and $q \in \mathbb{N}$ is a fixed number),

$$\begin{aligned} y_n &= P(\overline{\alpha}_n x_n + \overline{\beta}_n T_2(PT_2)^{n-1}x_n + \overline{\gamma}_n v_n), \quad n = 0, 1, 2, \ldots, \\ x_{n+1} &= P(\alpha_n x_n + \beta_n T_1(PT_1)^{n-1}y_{n-q} + \gamma_n u_n), \quad n = q, q+1, q+2, \ldots, \end{aligned} \tag{1.15}$$

where $T_1, T_2 : C \to X$ are nonself generalized asymptotically quasi-nonexpansive mappings, $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\overline{\alpha}_n\}$, $\{\overline{\beta}_n\}$ and $\{\overline{\gamma}_n\}$ are real sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = \overline{\alpha}_n + \overline{\beta}_n + \overline{\gamma}_n = 1$ for all $n \geq 0$. They gave the following strong convergence theorem.

Theorem 1.0.4 (see [19]). *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X, $T_1, T_2 : C \to X$ two uniformly L-Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings with nonnegative real sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$. Suppose $F = F(T_1) \cap F(T_2) \neq \emptyset$.*

For any $x_i \in C$ $(i = 0, 1, 2, \dots, q$ *and* $q \in \mathbb{N}$ *is a fixed number), let* $\{x_n\}$ *be the sequence defined by (1.15) satisfying* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $0 < \liminf_{n\to\infty} \overline{\alpha}_n \leq \limsup_{n\to\infty} \overline{\alpha}_n < 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \overline{\gamma}_n < \infty$. *If* T_1, T_2 *satisfies condition* A' *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a common fixed point of* T_1, T_2.

Recently, a new iterative scheme which is called the projection type Ishikawa iteration for two nonself asymptotically nonexpansive mappings was defined and constructed by Thianwan [48]. It is given as follows:

$$\begin{aligned} y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} &= P((1-\alpha_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n), \quad n \geq 1, \end{aligned} \tag{1.16}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $[0, 1)$. He studied the scheme for two nonself asymptotically nonexpansive mappings and proved strong convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to a common fixed point of T_1, T_2 under suitable conditions in a uniformly convex Banach space.

Note that Thianwan process (1.16) and Wang process (1.12) are independent: neither reduces to the other.

If $T_1 = T_2$ and $\beta_n = 0$ for all $n \geq 1$, then (1.16) reduces to (1.11). It also can be reduces to Schu process (1.3).

We note that, in applications, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. This leads us, in this paper, to introduce and study a new class of two-step iterative scheme with perturbations for solving the fixed point problem for nonself generalized asymptotically quasi-nonexpansive mappings. This iterative scheme can be viewed as an extension for Ishikawa type iterative schemes of Thianwan [48]. The scheme is defined as follows.

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2 : C \to X$ are given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} y_n &= P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n), \\ x_{n+1} &= P((1-\alpha_n-\lambda_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n + \lambda_n u_n), \quad n \geq 1, \end{aligned} \tag{1.17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are appropriate real sequences in $[0, 1)$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. We then prove its strong convergence under some suitable conditions in Banach spaces.

Note that Deng and Liu process (1.15) and our process (1.17) are independent: neither reduces to the other.

If $\gamma_n = \lambda_n = 0$ for all $n \geq 1$, then (1.17) reduces to (1.16). Now, we recall some well known concepts and results.

Fixed point theory is an immensely active area of research due to its applications in multiple fields. It addresses the results which state that, under certain conditions, a self map on a set admits a fixed point. Among all the results in fixed point theory, the Banach contraction principle (see [6]) in metric fixed point theory is the most celebrated one due to its simplicity and ease of application in major areas of mathematics. Following the Banach contraction principle, Boyd and Wong [8] investigated the fixed point results in nonlinear contraction mappings. Subsequently, many authors extended and generalized this fixed point theorem in different directions, in particular, by Reich [32]. In 2008, by combination of the concepts in fixed point theory and graph theory, Jachymski [22] generalized the Banach contraction principle in a complete metric space endowed with a directed graph. In 2012, Aleomraninejad et al. [5] presented some iterative scheme for G-contraction and G-nonexpansive mappings in a Banach space with a graph. In 2015, Alfuraidan and Khamsi [2] defined the concept of G-monotone nonexpansive multivalued mappings defined on a hyperbolic metric space with a graph. Alfuraidan [1] studied the existence of fixed points of monotone nonexpansive mappings on a Banach space endowed with a directed graph. Tiammee et al. [50] proved Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. They also proved the strong convergence of the Halpern iteration for a G-nonexpansive mapping.

In 2016, Tripak [49] introduced and studied the following Ishikawa iteration process:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_2 y_n \end{aligned} \tag{1.18}$$

to approximate common fixed points of two G-nonexpansive mappings in a Banach space endowed with a graph.

Agarwal et al. [3] introduced and studied the S-iteration process for a class of nearly asymptotically nonexpansive mappings in Banach spaces. They showed that this process has a better convergence rate than Ishikawa iteration for a class of contractions in metric spaces. Recently, an iterative scheme which is called the modified S-iteration for two G-nonexpansive mappings was defined and constructed by Suparatulatorn et al. [42]. It is given as follows:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} &= (1-\alpha_n)T_1 x_n + \alpha_n T_2 y_n,\ n \geq 0, \end{aligned} \tag{1.19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in $(0,1)$. They studied the strong and weak convergence of the iterative scheme (1.19) under proper conditions in a uniformly convex Banach space endowed with a graph.

Motivated by the recent works, we introduce and study a new two-step iteration process for two G-nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_0 \in C$ and

$$y_n = (1-\beta_n)x_n + \beta_n T_1 x_n,$$
$$x_{n+1} = (1-\alpha_n)T_1 y_n + \alpha_n T_2 y_n, \ n \geq 0, \tag{1.20}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate real sequences in (0, 1).

The first purpose of this dissertation is to introduce and study a new type of two-step iterative scheme which is called the projection type Ishikawa iteration with perturbations for two nonself generalized asymptotically quasi-nonexpansive mappings in Banach spaces. A sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is also established in a real uniformly convex Banach space. Furthermore, the strong convergence of a new iterative scheme with perturbations to a common fixed point of two nonself generalized asymptotically quasi-nonexpansive mappings on a nonempty closed convex subset of a real Banach space is proved.

The second purpose is to construct an iteration process for approximating common fixed points of two G-nonexpansive mappings and to prove some weak and strong convergence theorems for such mappings in a uniformly convex Banach space endowed with a graph. We also shows the numerical experiment for supporting our main results and comparing rate of convergence of the proposed method (1.20) with the Ishikawa iteration process (1.1) and the modified S-iteration process (1.2).

The thesis is divided into 4 chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with basic concepts and preliminaries and give some useful results that will be used in later chapters. Chapter 3 is the main results of this research with divided into two section as follows:

(1) Projection type Ishikawa iteration with perturbations for common fixed points of two nonself generalized asymptotically quasi-nonexpansive mappings.

(2) A new two-step iteration method for G-nonexpansive mappings in Banach spaces with a graph.

Chapter 4 summarizes all the theorems in this thesis.

CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

2.1 Metric spaces and Banach spaces

Now, we recall some well known concepts and results.

Definition 2.1.1. [25] A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$(triangle inequality).

Definition 2.1.2. [25] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or, simple} x_n \to x$$

we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Lemma 2.1.3. *[47] Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \ \forall n = 1, 2, ...,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(1) $\lim_{n\to\infty} a_n$ exists.

(2) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Definition 2.1.4. [25] A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if for every $\epsilon > 0$ there is an $N(\epsilon) \in N$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$.

Definition 2.1.5. [25] A metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.6. [25] Every convergent sequence in a matric space is a Cauchy sequence.

Theorem 2.1.7. *[28] Let* $\{x_n\}$ *be a sequence in* $\mathbb{R}$. *If every subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *has a convergent subsequence, then* $\{x_n\}$ *is convergent.*

Definition 2.1.8. [28] Let X be a matric space and A be any nonempty subset of X. For each x in X, the distance $d(x, A)$ from x to A is $\inf\{d(x, y)|y \in A\}$.

Definition 2.1.9. [28] Let X be a linear space (or vector space). A norm on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha|\|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a normed space or normed vector space or normed linear space.

Definition 2.1.10. [28] Let X be normed space. The metric induced by the norm of X is the metric d on X defined by the formula $d(x, y) = \|x - y\|$ for all $x, y \in X$. The norm topology of X is the topology obtained from this metric.

Definition 2.1.11. [28] A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a Banach space or B-space or complete normed space if its norm is a Banach norm.

2.2 Fixed points of nonexpansive, asymptotically nonexpansive and G-nonexpansive mappings

Definition 2.2.1. [54] Let C be subset of a Banach space X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T) = \{x \in C | x = Tx\}$.

Definition 2.2.2. [54] Let C be subset of a Banach space X. A self-mapping $f : C \to C$ is called contraction on C of there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contraction on C.

Theorem 2.2.3. *[43] (The Banach contraction principle)*
Let X be complete metric space and let f be a contraction of X. Then f has a unique fixed point.

Definition 2.2.4. [9] A mapping $T : C \to X$ is called demiclosed with respect to y if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \to x$ weakly and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 2.2.5. *[9] Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed point of T.*

Lemma 2.2.6. *[13] Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

Lemma 2.2.7. *([12], Theorem 3.4) Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and let $T : C \to X$ be an asymptotically* *nonexpansive mapping with a sequence* $\{k_n\} \subset [1,\infty)$ *and* $k_n \to 1$ *as* $n \to \infty$. *Then* $I - T$ *is demiclosed at zero. i.e., if* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixed point of* T.

Lemma 2.2.8. *[20] Suppose two mappings* $S, T : C \to C$, *where* C *is a subset of a normed space* X, *said to be satisfy condition* (A') *if there exists a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$, $f(r) > 0$ *for all* $r \in (0,\infty)$ *such that either* $\|x - Tx\| \geq f(d(x,F))$ *or* $\|x - Tx\| = f(d(x,F))$ *for all* $x \in C$ *where* $d(x,F)= inf\ \{\|x - p\| : p \in F = F(S) \bigcap F(T)\}$.

Lemma 2.2.9. *([44], Lemma 1) Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequence of nonnegative real numbers satisfying the inequality.*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ *and* $\sum_{n=1}^{\infty}b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. In particular, if* $\{a_n\}$ *has a subsequence converging to 0, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.10. *[34] Suppose that* X *be a uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X *such that* $\limsup_{n\to\infty}\|y_n\| \leq r$ *and* $\limsup_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ *hold for some* $r \geq 0$. *Then* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.2.11. *[41] Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X . *Let* $u, v \in X$ *be such that* $\lim_{n\to\infty}\|x_n - u\|$ *and* $\lim_{n\to\infty}\|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to* u *and* v, *respectively, then* $u = v$.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \epsilon = \|x - y\|\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset C of X is said to be a retract if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, the range of P.

A set C is optimal if each point outside C can be moved to be closer to all points of C. It is well known (see [18]) that

(1) If X is a separable, strictly convex, smooth, reflexive Banach space, and if $C \subset$X is an optimal set with interior, then C is a nonexpansive retract of X.

(2) A subset of l^p, with $1 < p < \infty$, is a nonexpansive retract if and only if it is optimal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. Moreover, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Recall that two mappings $S, T : C \to X$, where C is a subset of a normed space X, are said to satisfy condition A' (see [20]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either

$$\|x - Sx\| \geq f(d(x, F)) \quad \text{or} \quad \|x - Tx\| \geq f(d(x, F))$$

for all $x \in C$, where $d(x, F) = \inf\{\|x - q\| : q \in F = F(S) \cap F(T)\}$.

Note that condition A' reduces to condition (A) (see [47]) when $S = T$. Maiti and Ghosh [26] and Tan and Xu [47] have approximated fixed points of a nonexpansive mapping T by Ishikawa iterates under the condition (A).

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.2.12 (see [47])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ exists;

(ii) In particular, if $\{a_n\}$ has a sequence $\{a_{n_k}\}$ converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.13 (see [36])**.** *Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty subset of a real Banach space X. Let $\triangle$ denote the diagonal of the cartesian product $C \times C$, *i.e.*, $\triangle = \{(x, x) : x \in C\}$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \triangle$. We assume G has no parallel edge. So we can identify the graph G with the pair $(V(G), E(G))$. By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

We recall a few basic notions concerning the connectivity of graphs. All of them can be found, e.g., in [23]. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \ldots N-1$. A graph G is connected if there is a path between any two vertices. A directed graph $G = (V(G), E(G))$ is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we have $(x, z) \in E(G)$.

Let $x_0 \in V(G)$ and A a subset of $V(G)$. We say that A is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in A$. A dominates x_0 if for each $x \in A$, $(x, x_0) \in E(G)$.

We say that a mapping $T : C \to C$ is said to be G-contraction if T satisfies the following conditions:

(i) T preserves edges of G (or T is edge-preserving), i.e.,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G),$$

(ii) T decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|Tx - Ty\| \leq \alpha \|x - y\|.$$

A mapping $T : C \to C$ is said to be G-nonexpansive (see [2], Definition 2.3 (iii)) if T satisfies the following conditions:

(i) T preserves edges of G, i.e.,

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G),$$

(ii) T non-increases weights of edges of G in the following way:

$$(x, y) \in E(G) \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

A mapping $T : C \to C$ is said to be G-demiclosed at 0 if, for any sequence $\{x_n\}$ in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup x$ and $Tx_n \to 0$ imply $Tx = 0$.

A Banach space X is said to satisfy Opial's condition [29] if $x_n \rightharpoonup x$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. Recall that the mappings $T_i (i = 1, 2)$ on C are said to satisfy condition (B) [39] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that for all $x \in C$,

$$max\{\|x - T_1 x\|, \|x - T_2 x\|\} \geq f(d(x, F)),$$

where $F = F(T_1) \cap F(T_2)$, $F(T_i)(i = 1, 2)$ are the sets of fixed points of T_i and $d(x, F) = \inf\{\|x - q\| : q \in F\}$.

Let C be a subset of a metric space (X, d). A mapping $T : C \to C$ is semi-compact [39] if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$.

Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then, C is said to have Property $WG(SG)$ if for each sequence $\{x_n\}$ in C converging weakly (strongly) to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, x) \in E(G)$ for all $j \in \mathbb{N}$.

Remark 2.2.14 (see [42])**.** If G is transitive, then property WG is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in X, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.2.15 ([42])**.** *Suppose that X is a Banach space having Opial's condition, C has Property WG and let $T : C \to C$ be a G-nonexpansive mapping. Then, $I - T$ is G-demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.2.16 ([47])**.** *Let $\{a_n\}$ and $\{t_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + t_n \ \textit{for all}\ n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2.17 ([36])**.** *Let X be a uniformly convex Banach space, and $\{\alpha_n\}$ a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\limsup\limits_{n\to\infty} ||x_n|| \leq c, \limsup\limits_{n\to\infty} ||y_n|| \leq c$ and $\limsup\limits_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = c$ for some $c \geq 0$. Then $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

Lemma 2.2.18 ([41])**.** *Let X be a Banach space that satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim\limits_{n\to\infty} ||x_n - u||$ and $\lim\limits_{n\to\infty} ||x_n - v||$ exist. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.2.19 ([4])**.** *Let X be a uniformly convex Banach space, C be a nonempty bounded convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for any nonexpansive mapping $T : C \to X$, any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{\lambda_i\}_{i=1}^n$ with $\sum\limits_{i=1}^{n} \lambda_i = 1$, the following inequality holds:*

$$\gamma\|T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i T x_i\| \leq \max_{1\leq i,j\leq n} (\|x_i - x_j\| - \|Tx_i - Tx_j\|).$$

Lemma 2.2.20 ([35]). *Let* $\{x_n\}$ *be a bounded sequence in a reflexive Banach space* X. *If for any weakly convergent subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$, *both* $\{x_{n_j}\}$ *and* $\{x_{n_j+1}\}$ *converge weakly to the same point in* X, *then the sequence* $\{x_n\}$ *is weakly convergent.*

CHAPTER III

MAIN RESULTS

3.1 Projection type Ishikawa iteration with perturbations for common fixed points of two nonself generalized asymptotically quasi-nonexpansive mappings

Lemma 3.1.1. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in $[0, 1)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.17). If $q \in F$, then $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in F$, by boundedness of the sequences $\{u_n\}$ and $\{v_n\}$, so we can put

$$M = \max\{\sup_{n\geq 1} \|u_n - q\|, \sup_{n\geq 1} \|v_n - q\|\}.$$

Setting $k_n^{(1)} = 1 + r_n^{(1)}$, $k_n^{(2)} = 1 + r_n^{(2)}$. Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ $(i = 1, 2)$, so $\sum_{n=1}^{\infty} r_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} r_n^{(2)} < \infty$. Using (1.17), we have

$$\begin{aligned}
\|y_n - q\| &= \|P((1 - \beta_n - \gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n) - P(q)\| \\
&\leq \|(1 - \beta_n - \gamma_n)(x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q) + \gamma_n(v_n - q)\| \\
&\leq (1 - \beta_n - \gamma_n)\|x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}x_n - q\| + \gamma_n\|v_n - q\| \\
&\leq (1 - \beta_n - \gamma_n)\|x_n - q\| + \beta_n(1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M \\
&= (1 - \beta_n - \gamma_n)\|x_n - q\| + (\beta_n + \beta_n r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M \\
&\leq \|x_n - q\| + r_n^{(2)}\|x_n - q\| + \delta_n^{(2)} + \gamma_n M
\end{aligned}$$

$$= (1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M,$$

and so

$$\begin{aligned}
\|x_{n+1} - q\| &= \|P((1-\alpha_n-\lambda_n)y_n + \alpha_n T_1(PT_1)^{n-1}y_n + \lambda_n u_n) - P(q)\| \\
&\leq \|(1-\alpha_n-\lambda_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q) + \lambda_n(u_n - q)\| \\
&\leq (1-\alpha_n-\lambda_n)\|y_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - q\| + \lambda_n\|u_n - q\| \\
&\leq (1-\alpha_n-\lambda_n)\|y_n - q\| + \alpha_n(1 + r_n^{(1)})\|y_n - q\| + \delta_n^{(1)} + \lambda_n M \\
&= (1-\alpha_n-\lambda_n)\|y_n - q\| + (\alpha_n + \alpha_n r_n^{(1)})\|y_n - q\| + \delta_n^{(1)} + \lambda_n M \\
&\leq \|y_n - q\| + r_n^{(1)}\|y_n - q\| + \delta_n^{(1)} + \lambda_n M \\
&= (1 + r_n^{(1)})\|y_n - q\| + \delta_n^{(1)} + \lambda_n M \\
&\leq (1 + r_n^{(1)})((1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M) + \delta_n^{(1)} + \lambda_n M \\
&= (1 + r_n^{(1)})(1 + r_n^{(2)})\|x_n - q\| \\
&\quad + (1 + r_n^{(1)})\delta_n^{(2)} + (1 + r_n^{(1)})\gamma_n M + \delta_n^{(1)} + \lambda_n M \\
&= (1 + r_n^{(1)} + r_n^{(2)} + r_n^{(1)}r_n^{(2)})\|x_n - q\| + \varepsilon_n^{(1)},
\end{aligned}$$

where $\varepsilon_n^{(1)} = (1 + r_n^{(1)})\delta_n^{(2)} + (1 + r_n^{(1)})\gamma_n M + \delta_n^{(1)} + \lambda_n M$ and we note here that $\sum_{n=1}^{\infty} \epsilon_n^{(1)} < \infty$ since $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} r_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(1)} < \infty$ and $\sum_{n=1}^{\infty} \delta_n^{(2)} < \infty$. Since $\sum_{n=1}^{\infty}(r_n^{(1)} + r_n^{(2)} + r_n^{(1)}r_n^{(2)}) < \infty$ we obtained by Lemma 2.2.12 (i) that $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Lemma 3.1.2. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two uniformly $L-$Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1,\infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0,1)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. From an arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.17). Then $\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0$.*

Proof. Let $q \in F$. Setting $k_n^{(1)} = 1 + r_n^{(1)}$, $k_n^{(2)} = 1 + r_n^{(2)}$. By Lemma 3.1.1, we see that $\lim_{n\to\infty} \|x_n - q\|$ exists. It follows that $\{x_n\}$ and $\{y_n\}$ are bounded. Also, $\{u_n - y_n\}$ and $\{v_n - x_n\}$ are bounded. Now we set

$$C = \max\{\sup_{n\geq 1} \|u_n - y_n\|, \sup_{n\geq 1} \|v_n - x_n\|\}.$$

Assume that $\lim_{n\to\infty} \|x_n - q\| = c$. In addition,

$$\|y_n - q\| \leq (1 + r_n^{(2)})\|x_n - q\| + \delta_n^{(2)} + \gamma_n M, \tag{3.1}$$

where the notation M is taken from Lemma 3.1.1.

Taking the lim sup on both sides in the inequality (3.1), we have

$$\limsup_{n\to\infty} \|y_n - q\| \leq c. \tag{3.2}$$

Note that $\|y_n - q + \lambda_n(u_n - y_n)\| \leq \|y_n - q\| + \lambda_n C$ gives that

$$\limsup_{n\to\infty} \|y_n - q + \lambda_n(u_n - y_n)\| \leq c. \tag{3.3}$$

In addition, $\|T_1(PT_1)^{n-1}y_n - q + \lambda_n(u_n - y_n)\| \leq k_n^{(1)}\|y_n - q\| + \delta_n^{(1)} + \lambda_n C$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_1(PT_1)^{n-1}y_n - q + \lambda_n(u_n - y_n)\| \leq c. \tag{3.4}$$

In addition,

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|(1 - \alpha_n - \lambda_n)(y_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q) + \lambda_n(u_n - q)\| \\ &\leq (1 + r_n^{(1)} + r_n^{(2)} + r_n^{(1)}r_n^{(2)})\|x_n - q\| + \varepsilon_n^{(1)}, \end{aligned} \tag{3.5}$$

where the notation $\varepsilon_n^{(1)}$ is taken from Lemma 3.1.1.

Since $\sum_{n=1}^{\infty}(r_n^{(1)} + r_n^{(2)} + r_n^{(1)}r_n^{(2)}) < \infty$, $\sum_{n=1}^{\infty} \epsilon_n^{(1)} < \infty$ and $\lim_{n\to\infty} \|x_{n+1} - q\| = c$, letting $n \to \infty$ in the inequality (3.5), we have

$$\lim_{n\to\infty} \|(1-\alpha_n-\lambda_n)(y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)+\lambda_n(u_n-q)\| = c. \tag{3.6}$$

From

$$\|(1-\alpha_n)(y_n-q+\lambda_n(u_n-y_n))+\alpha_n(T_1(PT_1)^{n-1}y_n-q+\lambda_n(u_n-y_n))\| = \\ \|(1-\alpha_n-\lambda_n)(y_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)+\lambda_n(u_n-q)\|.$$

and (3.6), we have

$$\lim_{n\to\infty} \|(1-\alpha_n)(y_n-q+\lambda_n(u_n-y_n))+\alpha_n(T_1(PT_1)^{n-1}y_n-q+\lambda_n(u_n-y_n))\| \\ = c. \tag{3.7}$$

By using (3.3), (3.4), (3.7) and Lemma 2.2.13, we have

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}y_n-y_n\| = 0. \tag{3.8}$$

In addition,

$$\begin{aligned}\|T_2(PT_2)^{n-1}x_n-q+\gamma_n(v_n-x_n)\| &\le \|T_2(PT_2)^{n-1}x_n-q\|+\gamma_n\|v_n-x_n\| \\ &\le k_n^{(2)}\|x_n-q\|+\delta_n^{(2)}+\gamma_n C,\end{aligned}$$

and taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_2(PT_2)^{n-1}x_n-q+\gamma_n(v_n-x_n)\| \le c. \tag{3.9}$$

Using (1.17), we have

$$\begin{aligned}\|x_{n+1}-q\| &\le (1-\alpha_n-\lambda_n)\|y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\|+\lambda_n\|u_n-q\| \\ &= (1-\alpha_n-\lambda_n)\|y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-y_n+y_n-q\| \\ &\quad +\lambda_n\|u_n-y_n+y_n-q\|\end{aligned}$$

$$\leq (1-\alpha_n-\lambda_n)\|y_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-y_n\|$$
$$+\alpha_n\|y_n-q\|+\lambda_n\|u_n-y_n\|+\lambda_n\|y_n-q\|$$
$$\leq \|y_n-q\|+\|T_1(PT_1)^{n-1}y_n-y_n\|+\lambda_n C. \qquad (3.10)$$

Taking the lim inf on both sides in the inequality (3.10), by (3.8), $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\lim_{n\to\infty}\|x_{n+1}-q\|=c$, we have

$$\liminf_{n\to\infty}\|y_n-q\|\geq c. \qquad (3.11)$$

It follows from (3.2) and (3.11) that $\lim_{n\to\infty}\|y_n-q\|=c$. This implies that

$$c=\lim_{n\to\infty}\|y_n-q\|\leq \lim_{n\to\infty}\|(1-\beta_n-\gamma_n)(x_n-q)$$
$$+\beta_n(T_2(PT_2)^{n-1}x_n-q)+\gamma_n(v_n-q)\|$$
$$\leq \lim_{n\to\infty}\|x_n-q\|=c,$$

and so

$$\lim_{n\to\infty}\|(1-\beta_n-\gamma_n)(x_n-q)+\beta_n(T_2(PT_2)^{n-1}x_n-q)+\gamma_n(v_n-q)\|=c. \qquad (3.12)$$

From

$$\|(1-\beta_n)(x_n-q+\gamma_n(v_n-x_n))+\beta_n(T_2(PT_2)^{n-1}x_n-q+\gamma_n(v_n-x_n))\|$$
$$=\|(1-\beta_n-\gamma_n)(x_n-q)+\beta_n(T_2(PT_2)^{n-1}x_n-q)+\gamma_n(v_n-q)\|$$

and (3.12), we have

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n-q+\gamma_n(v_n-x_n))+\beta_n(T_2(PT_2)^{n-1}x_n-q+\gamma_n(v_n-x_n))\|=c. \qquad (3.13)$$

Note that $\|x_n-q+\gamma_n(v_n-x_n)\|\leq\|x_n-q\|+\gamma_n C$ gives that

$$\limsup_{n\to\infty}\|x_n-q+\gamma_n(v_n-x_n)\|\leq c. \qquad (3.14)$$

Using (3.9), (3.13), (3.14) and Lemma 2.2.13, we obtain

$$\lim_{n\to\infty} \|T_2(PT_2)^{n-1}x_n - x_n\| = 0. \tag{3.15}$$

From $y_n = P((1-\beta_n-\gamma_n)x_n+\beta_n T_2(PT_2)^{n-1}x_n+\gamma_n v_n)$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and (3.15), we have

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-\beta_n-\gamma_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n + \gamma_n v_n) - x_n\| \\
&\le \|(1-\beta_n-\gamma_n)(x_n - x_n) + \beta_n(T_2(PT_2)^{n-1}x_n - x_n) + \gamma_n(v_n - x_n)\| \\
&\le \beta_n\|T_2(PT_2)^{n-1}x_n - x_n\| + \gamma_n\|v_n - x_n\| \\
&\le \|T_2(PT_2)^{n-1}x_n - x_n\| + \gamma_n C \\
&\to 0 \quad (\text{as } n\to\infty).
\end{aligned} \tag{3.16}$$

Now, since $T_i\,(i=1,2)$ are uniformly L-Lipschitzian for Lipschitz constant $L = \max_{1\le i\le 2}\{L_i\} > 0$. We note that

$$\begin{aligned}
\|T_1(PT_1)^{n-1}x_n - x_n\| &= \|T_1(PT_1)^{n-1}x_n - y_n + y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - y_n\| + \|y_n - x_n\| \\
&= \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n \\
&\quad + T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_n\| \\
&\quad + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\| \\
&\le L\|x_n - y_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \|y_n - x_n\|.
\end{aligned}$$

Thus, it follows from (3.8) and (3.16) that

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \tag{3.17}$$

By using (1.17), we have

$$\|x_{n+1} - x_n\| \le (1-\alpha_n-\lambda_n)\|y_n - x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - x_n\| + \lambda_n\|u_n - x_n\|$$

$$
\begin{aligned}
&\leq (1-\alpha_n-\lambda_n)\|y_n-x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n + y_n - x_n\| \\
&\quad + \lambda_n\|u_n - y_n + y_n - x_n\| \\
&\leq (1-\alpha_n-\lambda_n)\|y_n-x_n\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - y_n\| \\
&\quad + \alpha_n\|y_n - x_n\| + \lambda_n\|u_n - y_n\| + \lambda_n\|y_n - x_n\| \\
&\leq \|y_n - x_n\| + \|T_1(PT_1)^{n-1}y_n - y_n\| + \lambda_n C.
\end{aligned}
$$

It follows from (3.8), (3.16) and $\sum_{n=1}^{\infty}\lambda_n < \infty$ that

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.18}
$$

Using (3.17) and (3.18), we have

$$
\begin{aligned}
\|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-1}x_n \\
&\quad + T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\
&\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + L\|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\|, \\
&\to 0 \quad (\text{as } n\to\infty).
\end{aligned} \tag{3.19}
$$

In addition, for $n \geq 2$,

$$
\begin{aligned}
\|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| &= \|x_{n+1} - x_n + x_n - T_1(PT_1)^{n-2}x_n \\
&\quad + T_1(PT_1)^{n-2}x_n - T_1(PT_1)^{n-2}x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\quad + \|T_1(PT_1)^{n-2}x_{n+1} - T_1(PT_1)^{n-2}x_n\| \\
&\leq \|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-2}x_n - x_n\| \\
&\quad + L\|x_{n+1} - x_n\|.
\end{aligned}
$$

It follows from (3.18) and (3.19) that

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-2}x_{n+1}\| = 0. \tag{3.20}$$

We denote as $(PT_1)^{1-1}$ the identity maps from C onto itself. Thus by the inequality (3.19) and (3.20), we have

$$\begin{aligned}
\|x_{n+1} - T_1x_{n+1}\| &= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1} + T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + \|T_1(PT_1)^{n-1}x_{n+1} - T_1x_{n+1}\| \\
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|(PT_1)^{n-1}x_{n+1} - x_{n+1}\| \\
&= \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|(PT_1)(PT_1)^{n-2}x_{n+1} - P(x_{n+1})\| \\
&\le \|x_{n+1} - T_1(PT_1)^{n-1}x_{n+1}\| + L\|T_1(PT_1)^{n-2}x_{n+1} - x_{n+1}\| \\
&\to 0 \quad (\text{as } n \to \infty),
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0$. Similary, we may show that

$$\lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

The proof is completed. □

We prove the strong convergence of the scheme (1.17) under condition A' which is weaker than the compactness of the domain of the mappings.

Theorem 3.1.3. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two uniformly $L-$Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1,\infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0,1)$ such that*

$\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.17) converge strongly to a common fixed point of T_1 and T_2.

Proof. By Lemma 3.1.2, we have $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0$. It follows from condition A' that

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty}\|x_n - T_1x_n\| = 0 \quad \text{or}$$

$$\lim_{n\to\infty} f(d(x_n, F)) \le \lim_{n\to\infty}\|x_n - T_2x_n\| = 0.$$

In the both case, $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. That is

$$\lim_{n\to\infty}\inf_{y^*\in F}\|x_n - y^*\| = \lim_{n\to\infty} d(x_n, F) = 0.$$

It implies that

$$\inf_{y^*\in F}\lim_{n\to\infty}\|x_n - y^*\| = 0.$$

So, for any given $\varepsilon > 0$, there exists $p \in F$ and $N > 0$ such that for all $n \ge N$ $\|x_n - p\| < \frac{\varepsilon}{2}$. This shows that

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p\| + \|x_n - p\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for all $n \ge N$ and $m \ge 1$. Hence, $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u$. From $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0$ and the continuity of T_1 and T_2, we have $\|T_1u - u\| =$

$\|T_2u - u\| = 0$. Thus $u \in F$. From (3.16), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,$$

it follows that $\lim_{n\to\infty} \|y_n - u\| = 0$. This completes the proof. □

The following result follows from Theorem 4.1.1.

Theorem 3.1.4. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself asymptotically nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ $(i = 1, 2)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0,1)$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.17) converge strongly to a common fixed point of T_1 and T_2.*

For $\gamma_n = \lambda_n = 0$, the iterative scheme (1.17) reduces to that of (1.16) for uniformly L-Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings and the following result is directly obtained by Theorem 4.1.1.

Theorem 3.1.5. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two uniformly $L-$Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1,\infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty}\delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.16) converge strongly to a common fixed point of T_1 and T_2.*

The following result follows from Theorem 4.1.3.

Theorem 3.1.6. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself asymptotically nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.16) converge strongly to a common fixed point of T_1 and T_2.*

In the remainder of this section, we deal with the strong convergence of the new iterative scheme (1.17) to a common fixed point of nonself generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Theorem 3.1.7. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$ is closed. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0, 1)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequence $\{x_n\}$ defined by the iterative scheme (1.17) converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{y \in F} \|x_n - y\|$, $n \geq 1$.*

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. As in the proof of Lemma 3.1.1, by the arbitrariness of $q \in F$, we have

$$\|x_{n+1}-q\| \leq (1+r_n^{(1)}+r_n^{(2)}+r_n^{(1)}r_n^{(2)})\|x_n-q\|+\varepsilon_n^{(1)},$$

and so

$$d(x_{n+1},F) \leq (1+r_n^{(1)}+r_n^{(2)}+r_n^{(1)}r_n^{(2)})d(x_n,F)+\varepsilon_n^{(1)},$$

where $\varepsilon_n^{(1)} = (1+r_n^{(1)})\delta_n^{(2)} + (1+r_n^{(1)})\gamma_n M + \delta_n^{(1)} + \lambda_n M$. Since $\sum_{n=1}^{\infty}(r_n^{(1)} + r_n^{(2)} + r_n^{(1)}r_n^{(2)}) < \infty$ and $\sum_{n=1}^{\infty}\epsilon_n^{(1)} < \infty$, we obtained by Lemma 2.2.12 that $\lim_{n\to\infty} d(x_n,F)$ exists. Then, by hypothesis $\liminf_{n\to\infty} d(x_n,F) = 0$, we have $\lim_{n\to\infty} d(x_n,F) = 0$. From Theorem 4.1.1, it obtain that $\{x_n\}$ defined by (1.17) is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = u$. Now $\lim_{n\to\infty} d(x_n,F) = 0$ gives that $d(u,F) = 0$. F is closed; therefore $u \in F$. This completes the proof of Theorem 4.1.5. $\square$

3.2 A new two-step iteration method for G-nonexpansive mappings in Banach spaces with a graph

Throughout the section, we let C be a nonempty closed convex subset of a Banach space X endowed with a directed graph G such that $V(G) = C$ and $E(G)$ is convex. We also suppose that the graph G is transitive. The mappings T_i ($i = 1, 2$) are G-nonexpansive from C to C with $F = F(T_1) \cap F(T_2)$ nonempty. For an arbitrary $x_0 \in C$, defined the sequence $\{x_n\}$ by (1.20)

We start with proving the following useful results.

Proposition 3.2.1. *Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0)$ are in $E(G)$. Then $(x_n, z_0), (y_n, z_0), (z_0, x_n), (z_0, y_n), (x_n, y_n)$ and (x_n, x_{n+1}) are in $E(G)$.*

Proof. We proceed by induction. Since T_1 is edge-preserving and $(x_0, z_0) \in E(G)$, we have $(T_1x_0, z_0) \in E(G)$ and so $(y_0, z_0) \in E(G)$, by $E(G)$ is convex. Again, by edge-preserving of T_1 and $(y_0, z_0) \in E(G)$, we have $(T_1y_0, z_0) \in E(G)$. Then, since T_2 is edge-preserving and $(y_0, z_0) \in E(G)$, we get $(T_2y_0, z_0) \in E(G)$. By the convexity of $E(G)$ and $(T_1y_0, z_0,), (T_2y_0, z_0) \in E(G)$, we get $(x_1, z_0) \in E(G)$. Thus, by edge-preserving of T_1, $(T_1x_1, z_0) \in E(G)$. Again, by the convexity of $E(G)$ and $(T_1x_1, z_0,), (x_1, z_0) \in E(G)$, we have $(y_1, z_0) \in E(G)$ and hence, (T_1y_1, z_0) and $(T_2y_1, z_0) \in E(G)$. Next, we assume that $(x_k, z_0) \in E(G)$. Since T_1 is edge-preserving, we get $(T_1x_k, z_0) \in E(G)$ and hence, $(y_k, z_0) \in E(G)$, since $E(G)$ is convex. Hence, by edge-preserving of T_1 and $(y_k, z_0) \in E(G)$, we have $(T_1y_k, z_0) \in E(G)$. Since T_2 is edge-preserving, we have $(T_2y_k, z_0) \in E(G)$. By the convexity of $E(G)$, we get $(x_{k+1}, z_0) \in E(G)$. Hence, by edge-preserving of T_1, we obtain $(T_1x_{k+1}, z_0) \in E(G)$, and so $(y_{k+1}, z_0) \in E(G)$, since $E(G)$ is convex. Therefore, $(x_n, z_0), (y_n, z_0) \in E(G)$ for all $n \geq 1$. Since T_1 is edge-preserving and $(z_0, x_0) \in E(G)$, we have $(z_0, T_1x_0) \in E(G)$, and so $(z_0, y_0) \in$

$E(G)$. Using a similar argument, we can show that $(z_0, x_n), (z_0, y_n) \in E(G)$ under the assumption that $(z_0, x_0) \in E(G)$ and $(z_0, y_0) \in E(G)$. By the transitivity of G, we get $(x_n, y_n), (x_n, x_{n+1}) \in E(G)$. This completes the proof. □

Lemma 3.2.2. *Let X be a uniformly convex Banach space. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$. Then*

(i) $\lim_{n\to\infty} \|x_n - z_0\|$ *exists;*

(ii) $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|$.

Proof. (i) Let $z_0 \in F$. By Proposition 3.2.1, we have $(x_n, z_0), (y_n, z_0) \in E(G)$. Then, by G-nonexpansiveness of $T_i (i = 1, 2)$ and using (1.20), we have

$$\begin{aligned}
\|y_n - z_0\| &= \|(1-\beta_n)x_n + \beta_n T_1 x_n - z_0\| \\
&= \|(1-\beta_n)(x_n - z_0) + \beta_n (T_1 x_n - z_0)\| \\
&\leq (1-\beta_n)\|x_n - z_0\| + \beta_n \|T_1 x_n - z_0\| \\
&\leq (1-\beta_n)\|x_n - z_0\| + \beta_n \|x_n - z_0\| \\
&= \|x_n - z_0\|, \qquad (3.21)
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - z_0\| &= \|(1-\alpha_n)T_1 y_n + \alpha_n T_2 y_n - z_0\| \\
&= \|(1-\alpha_n)(T_1 y_n - z_0) + \alpha_n (T_2 y_n - z_0)\| \\
&\leq (1-\alpha_n)\|T_1 y_n - z_0\| + \alpha_n \|T_2 y_n - z_0\| \\
&\leq (1-\alpha_n)\|y_n - z_0\| + \alpha_n \|y_n - z_0\| \\
&= \|y_n - z_0\| \\
&\leq \|x_n - z_0\|. \qquad (3.22)
\end{aligned}$$

It follows from Lemma 2.2.16 that $\lim_{n\to\infty}\|x_n - z_0\|$ exists. In particular, the sequence $\{x_n\}$ is bounded.

(ii) Assume that $\lim_{n\to\infty}\|x_n - z_0\| = c$. If $c = 0$, then by G-nonexpansiveness of $T_i(i = 1, 2)$, we get

$$\begin{aligned}\|x_n - T_i x_n\| &\leq \|x_n - z_0\| + \|z_0 - T_i x_n\| \\ &\leq \|x_n - z_0\| + \|z_0 - x_n\|.\end{aligned}$$

Therefore, the result follows. Suppose that $c > 0$. Taking the lim sup on both sides in the inequality (3.21), we obtain

$$\limsup_{n\to\infty}\|y_n - z_0\| \leq \limsup_{n\to\infty}\|x_n - z_0\| = c. \tag{3.23}$$

In addition, by G-nonexpansiveness of $T_i(i = 1, 2)$, we have $\|T_i y_n - z_0\| \leq \|y_n - z_0\|$, taking the lim sup on both sides in this inequality and using (3.23), we obtain

$$\limsup_{n\to\infty}\|T_i y_n - z_0\| \leq c. \tag{3.24}$$

Since $\lim_{n\to\infty}\|x_{n+1} - z_0\| = c$. Letting $n \to \infty$ in the inequality (3.22), we have

$$\lim_{n\to\infty}\|(1-\alpha_n)(T_1 y_n - z_0) + \alpha_n(T_2 y_n - z_0)\| = c. \tag{3.25}$$

By using (3.24), (3.25) and Lemma 2.2.17, we have

$$\lim_{n\to\infty}\|T_1 y_n - T_2 y_n\| = 0. \tag{3.26}$$

Note that $\|x_{n+1} - z_0\| \leq \|y_n - z_0\|$ gives that

$$\liminf_{n\to\infty}\|y_n - z_0\| \geq c. \tag{3.27}$$

From (3.23) and (3.27), we have

$$\lim_{n\to\infty} \|y_n - z_0\| = c. \tag{3.28}$$

From (3.21) and (3.28), we have

$$\lim_{n\to\infty} \|(1-\beta_n)(x_n - z_0) + \beta_n(T_1x_n - z_0)\| = c. \tag{3.29}$$

In addition, $\limsup_{n\to\infty} \|T_1x_n - z_0\| \leq \limsup_{n\to\infty} \|x_n - z_0\| = c$, using (3.29) and Lemma 2.2.17, we have

$$\lim_{n\to\infty} \|T_1x_n - x_n\| = 0. \tag{3.30}$$

Thus, it follows from (3.30) that

$$\begin{aligned}\|y_n - x_n\| &= \|(1-\beta_n)x_n + \beta_nT_1x_n - x_n\| \\ &\leq \beta_n\|T_1x_n - x_n\| \\ &\to 0 \ (as\ n \to \infty).\end{aligned} \tag{3.31}$$

Using (3.30), (3.31) together with G-nonexpansiveness of T_1, we have

$$\begin{aligned}\|T_1y_n - y_n\| &= \|T_1y_n - T_1x_n + T_1x_n - y_n\| \\ &\leq \|T_1y_n - T_1x_n\| + \|T_1x_n - y_n\| \\ &\leq \|y_n - x_n\| + \|T_1x_n - x_n\| + \|x_n - y_n\| \\ &\to 0 \ (as\ n \to \infty).\end{aligned} \tag{3.32}$$

Using (3.26), (3.31), (3.32) together with G-nonexpansiveness of T_2, we have

$$\begin{aligned}\|T_2x_n - x_n\| &= \|T_2x_n - y_n + y_n - x_n\| \\ &\leq \|T_2x_n - y_n\| + \|y_n - x_n\| \\ &= \|T_2x_n - T_2y_n + T_2y_n - y_n\| + \|y_n - x_n\|\end{aligned}$$

$$\leq \|T_2x_n - T_2y_n\| + \|T_2y_n - y_n\| + \|y_n - x_n\|$$
$$\leq \|x_n - y_n\| + \|T_2y_n - T_1y_n\| + \|T_1y_n - y_n\| + \|y_n - x_n\|$$
$$\to 0 \ (as\ n \to \infty).$$

Therefore, we conclude $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. This completes the proof. □

We now prove the weak convergence of the sequence generated by the new iteration process (1.20) for two G-nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.2.3. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C has Property WG. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0) \in E(G)$. From Lemma 3.2.2 (i), we have $\lim_{n\to\infty} \|x_n - z_0\|$ exists, so $\{x_n\}$ is bounded. It follows from Lemma 3.2.2 (ii) that $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup u$ as $n \to \infty$, without loss of generality. By Lemma 2.2.15, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. By Lemma 3.2.2 (ii), we obtain that $\|x_{n_k} - T_ix_{n_k}\| \to 0$ and $\|x_{n_j} - T_ix_{n_j}\| \to 0$ as $k, j \to \infty$. Using Lemma 2.2.15, we have $u, v \in F$. By Lemma 3.2.2 (i), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.2.18 that $u = v$. Therefore, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. □

Note that the Opial's condition has remained key to prove weak convergence theorems. However, each l^p $(1 \leq p < \infty)$ satisfies the Opial's condition, while all L^p do not have the property unless $p = 2$.

Next, we deal with the weak convergence of the sequence $\{x_n\}$ generated by (1.3) for two G-nonexpansive mappings without assuming the Opial's condition in a uniformly convex Banach space with a directed graph.

We start with proving the following lemma for later use.

Lemma 3.2.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and suppose that C has property WG. Let T be a G-nonexpansive mapping on C. Then $I - T$ is G-demiclosed at* 0.

Proof. Let $\{x_n\}$ be a sequence in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup q \in C$ and $(I - T)x_n \to 0$ as $n \to \infty$. By property WG, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, q) \in E(G)$ for all $j \in \mathbb{N}$. By Remark 2.2.14, $(x_n, q) \in E(G)$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ weakly converges in a uniformly convex Banach space X, it is bounded and hence there exists $r \geq 0$ such that $\{x_n\} \subset D = C \cap B(0, r)$. Then D is nonempty closed convex subset of C. Thus, $T : D \to C$ is G-nonexpansive mapping. By Mazur's theorem (see [51]), for each positive integer n, there exists a convex combination $y_n = \displaystyle\sum_{i=n}^{m(n)} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\displaystyle\sum_{i=n}^{m(n)} \lambda_i = 1$ such that

$$\|y_n - q\| < \frac{1}{n}. \tag{3.33}$$

Since $E(G)$ is convex and $(x_i, q) \in E(G)$ for each $i \in \mathbb{N}$, we must have that

$$\sum_{i=n}^{m(n)} \lambda_i(x_i, q) = (\sum_{i=n}^{m(n)} \lambda_i x_i, \sum_{i=n}^{m(n)} \lambda_i q)$$
$$= (\sum_{i=n}^{m(n)} \lambda_i x_i, q)$$
$$= (y_n, q) \in E(G).$$

It follows from $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ that for every $\epsilon > 0$, there exists a positive integer N such that

$$\|x_n - Tx_n\| < \epsilon \tag{3.34}$$

for every $n > N$. On the other hand, using Lemma 2.2.19 and (3.34), we have

$$\begin{aligned}
\|Ty_n - y_n\| &= \|Ty_n - \sum_{i=n}^{m(n)} \lambda_i Tx_i + \sum_{i=n}^{m(n)} \lambda_i Tx_i - \sum_{i=n}^{m(n)} \lambda_i x_i\| \\
&\leq \|Ty_n - \sum_{i=n}^{m(n)} \lambda_i Tx_i\| + \|\sum_{i=n}^{m(n)} \lambda_i Tx_i - \sum_{i=n}^{m(n)} \lambda_i x_i\| \\
&\leq \|Ty_n - \sum_{i=n}^{m(n)} \lambda_i Tx_i\| + \sum_{i=n}^{m(n)} \lambda_i \|Tx_i - x_i\| \\
&\leq \gamma^{-1} \max_{n\leq i,j\leq m(n)} (\|x_i - x_j\| - \|Tx_i - Tx_j\|) + \epsilon \\
&\leq \gamma^{-1} \max_{n\leq i,j\leq m(n)} (\|x_i - Tx_i\| + \|x_j - Tx_j\|) + \epsilon.
\end{aligned}$$

Therefore, from (3.33), (3.34), (3.35) and G-nonexpansiveness of T, we have

$$\begin{aligned}
\|q - Tq\| &\leq \|q - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tq\| \\
&\leq 2\|q - y_n\| + \gamma^{-1} \max_{n\leq i,j\leq m(n)} (\|x_i - Tx_i\| + \|x_j - Tx_j\|) + \epsilon \\
&\leq \frac{2}{n} + \gamma^{-1}(2\epsilon) + \epsilon,
\end{aligned}$$

for $n > N$. Taking lim sup on both sides in this inequality, we obtain

$$\|q - Tq\| \leq \gamma^{-1}(2\epsilon) + \epsilon. \tag{3.35}$$

Since γ is monotonically increasing with $\gamma(0) = 0$ and ϵ is arbitrary, we must have

$$\|q - Tq\| = 0. \tag{3.36}$$

Therefore, $q = Tq$. This completes the proof. □

Theorem 3.2.5. *Let X be a uniformly convex Banach space. Suppose that C has Property WG, $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, F is dominated by x_0 and F dominates x_0. If $(x_0, z_0), (z_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $z_0 \in F$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. Let $z_0 \in F$ be such that $(x_0, z_0), (z_0, x_0)$ are in $E(G)$. From Lemma 3.2.2 (i), we have $\lim_{n\to\infty} \|x_n - z_0\|$ exists, so $\{x_n\}$ is bounded in C. Since C is nonempty closed convex subset of a uniformly convex Banach space X, it is weakly compact and hence there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some point $p \in C$. By Lemma 3.2.2 (ii) we obtain that

$$\lim_{j\to\infty} \|x_{n_j} - T_1 x_{n_j}\| = 0 = \lim_{j\to\infty} \|x_{n_j} - T_2 x_{n_j}\|. \tag{3.37}$$

In addition, $\|T_2 y_n - y_n\| \leq \|T_2 y_n - T_1 y_n\| + \|T_1 y_n - y_n\|$, using (3.26) and (3.32), we have

$$\lim_{n\to\infty} \|T_2 y_n - y_n\| = 0. \tag{3.38}$$

Using Lemma 3.2.4, we have $I - T_1$ and $I - T_2$ are G-demiclosed at 0 so that $p \in F$. To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p.

To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 2.2.20. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $q \in C$. By similar arguments as above q is in F. Now for each $j \geq 1$, using (1.20), we have

$$x_{n_j+1} = (1-\alpha_{n_j})T_1y_{n_j} + \alpha_{n_j}T_2y_{n_j}. \tag{3.39}$$

It follows from (3.37) that

$$T_1x_{n_j} = (T_1x_{n_j} - x_{n_j}) + x_{n_j} \rightharpoonup q. \tag{3.40}$$

Now from (1.20) and (3.40),

$$y_{n_j} = (1-\beta_{n_j})x_{n_j} + \beta_{n_j}T_1x_{n_j} \rightharpoonup q. \tag{3.41}$$

Using (3.32) and (3.41), we have

$$T_1y_{n_j} = (T_1y_{n_j} - y_{n_j}) + y_{n_j} \rightharpoonup q. \tag{3.42}$$

Also from (3.38) and (3.41), we have

$$T_2y_{n_j} = (T_2y_{n_j} - y_{n_j}) + y_{n_j} \rightharpoonup q. \tag{3.43}$$

It follows from (3.39), (3.42) and (3.43) that

$$x_{n_j+1} \rightharpoonup q.$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 2.2.20 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$. This completes the proof. □

In the remainder of this section, we deal with the strong convergence of the sequence generated by the new iteration process (1.20) for two G-nonexpansive mappings in a uniformly convex Banach space with a directed graph.

Theorem 3.2.6. *Let X be a uniformly convex Banach space. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, $T_i (i = 1,2)$ satisfy condition (B), F is dominated by x_0 and F dominates x_0. Then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.2.2 (i), we have $\lim_{n\to\infty} \|x_n - q\|$ exists and so $\lim_{n\to\infty} d(x_n, F)$ exists for any $q \in F$. Also by Lemma 3.2.2 (ii), $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. It follows from condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, we obtain that $\lim_{n\to\infty} d(x_n, F) = 0$. Hence, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{u_j\} \subset F$ such that $\|x_{n_j} - u_j\| \leq \frac{1}{2^j}$. Put $n_{j+1} = n_j + k$ for some $k \geq 1$. Then

$$\|x_{n_{j+1}} - u_j\| \leq \|x_{n_j+k-1} - u_j\| \leq \|x_{n_j} - u_j\| \leq \frac{1}{2^j}.$$

We obtain that $\|u_{j+1} - u_j\| \leq \frac{3}{2^{j+1}}$, so $\{u_j\}$ is a Cauchy sequence. We assume that $u_j \to q_0 \in C$ as $j \to \infty$. Since F is closed, we get $q_0 \in F$. So we have $x_{n_j} \to q_0$ as $j \to \infty$. Since $\lim_{n\to\infty} \|x_n - q_0\|$ exists, we obtain $x_n \to q_0$. This completes the proof. □

Theorem 3.2.7. *Let X be a uniformly convex Banach space. Suppose that C has Property SG, $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$, F is dominated by x_0 and F dominates x_0. If one of $T_i (i = 1,2)$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. It follows from Lemma 3.2.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2x_n\|$. Since one of T_1 and T_2 is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$ as $j \to \infty$. Since C has Property SG and transitivity of graph G, we obtain $(x_{n_j}, q) \in E(G)$. Notice that, for each $i \in \{1,2\}$, $\lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$. Then

$$\begin{aligned} \|q - T_i q\| &\leq \|q - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|T_i x_{n_j} - T_i q\| \\ &\leq \|q - x_{n_j}\| + \|x_{n_j} - T_i x_{n_j}\| + \|x_{n_j} - q\| \\ &\to 0 \ \text{(as } j \to \infty). \end{aligned}$$

Hence $q \in F$. Thus $\lim_{n\to\infty} d(x_n, F)$ exists by Theorem 3.2.6. We note that $d(x_{n_j}, F) \leq d(x_{n_j}, q) \to 0$ as $j \to \infty$, hence $\lim_{n\to\infty} d(x_n, F) = 0$. It follows, as in the proof of Theorem 3.2.6, that $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2. This completes the proof. □

Now we are ready to discuss an example as well as the numerical experiments for supporting our main theorem. The following definitions will be useful in this context.

In 1976, Rhoades [33] gave the idea how to compare the rate of convergence between two iterative methods as follows:

Definition 3.2.8. [33] Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose that $\{x_n\}$ and $\{z_n\}$ are two iterations which converge to a fixed point q of T. Then $\{x_n\}$ is said to converge faster than $\{z_n\}$ if

$$\|x_n - q\| \leq \|z_n - q\|$$

for all $n \geq 1$.

In 2011, Phuengrattana and Suantai [31] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line.

In order to study, the order of convergence of a real sequence $\{a_n\}$ converging to a, we usually use the well-known terminology in numerical analysis, see [10], for example.

Definition 3.2.9. [10] Suppose $\{a_n\}$ is a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^{\alpha}} = \lambda,$$

then $\{a_n\}$ converges to a of order α, with asymptotic error constant λ. If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\alpha = 2$, the sequence is quadratically convergent. Berinde [7] employed above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 3.2.10. [7] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a and b, respectively. Assume that there exists

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}$ converges to a faster than the sequence $\{b_n\}$ to b.

(ii) If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Example 3.2.11. *Let $X = \mathbb{R}$ and $C = [0, 2]$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.50 \leq x, y \leq 1.70$. Define a mapping $T_1, T_2 : C \to C$ by*

$$T_1 x = \frac{2}{3} \arcsin(x - 1) + 1,$$

$$T_2x = \sqrt{x},$$

for any $x \in C$. It is easy to show that T_1, T_2 are G-nonexpansive but T_1, T_2 are not nonexpansive because

$$|T_1x - T_1y| > 0.50 = |x - y|$$

and

$$|T_2u - T_2v| > 0.45 = |u - v|$$

when $x = 1.95, y = 1.45, u = 0.5$ and $v = 0.05$. Let

$$\alpha_n = \frac{n+1}{5n+3}$$

and

$$\beta_n = \frac{n+2}{8n+5}.$$

Choose $z_0 = y_0 = x_0 = 1.35$. We note that $x = 1$ is a common fixed point of T_1 and T_2. Let $\{x_n\}$ be a sequence generated by (1.20) and $\{y_n\}$, $\{z_n\}$ be sequences generated by modified S-iteration and Ishikawa iteration, respectively. By computing, we obtain the following numerical experiments for common fixed point of T_1 and T_2 and rate of convergence of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$.

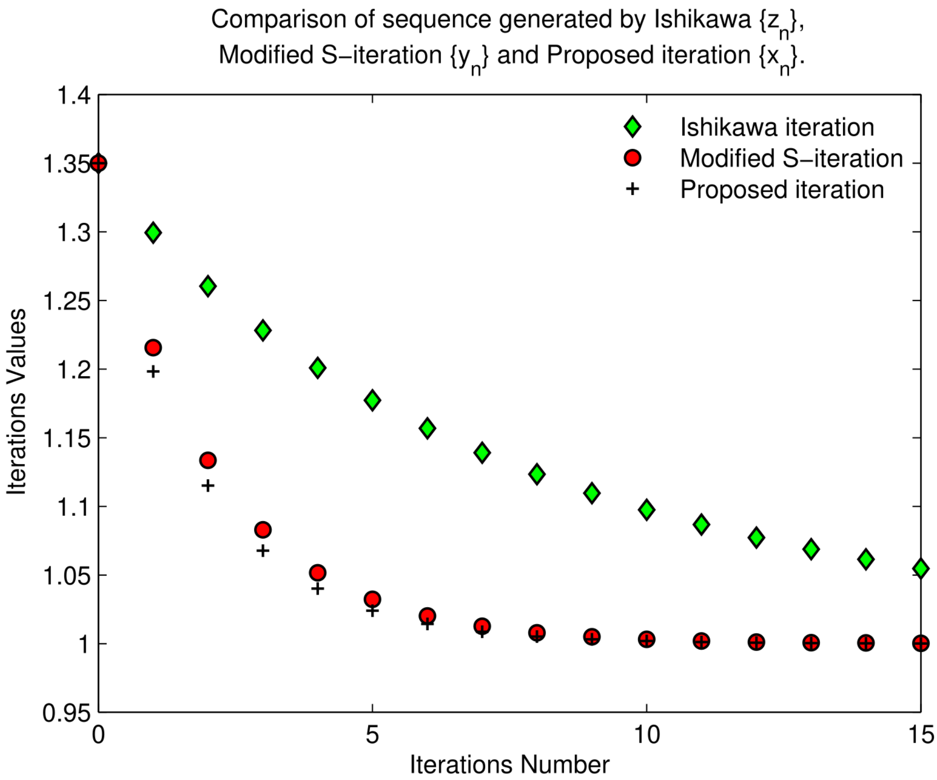

Figure 1: Numerical experiment of Example 3.2.11 by using Ishikawa iteration, modified S-iteration and the proposed method.

Figure 1 presents the three comparative methods consist of Ishikawa iteration, modified S-iteration and the proposed method converge to the solutions of the numerical experiment.

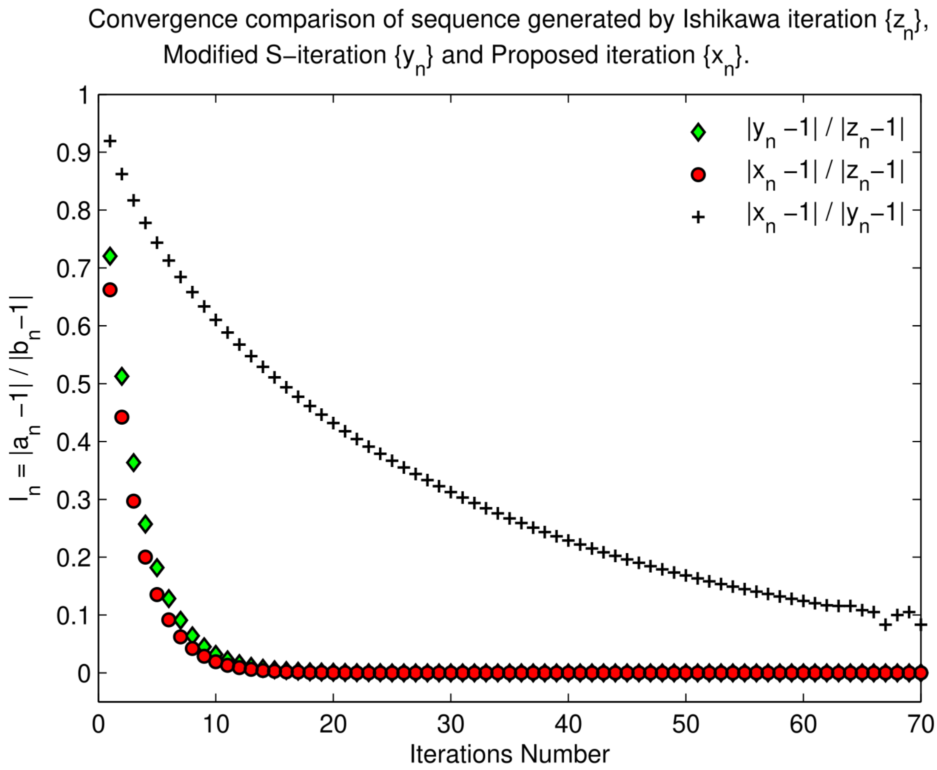

Figure 2: Convergence comparison of sequence generated by Ishikawa iteration, modified S-iteration and the proposed method for Example 3.2.11

Figure 2 shows the convergence comparison between two sequences $\{a_n\}$ and $\{b_n\}$ that converge to the same limit results from the parameter $l_n = \frac{|a_n-1|}{|b_n-1|}$ lead from Definition 3.2.10. The diamond plot shows the convergence comparison between two sequences generated by modified S-iteration and Ishikawa iteration, the circle plot shows the convergence comparison between two sequences generated by proposed method and Ishikawa iteration and the plus sign plot shows the convergence comparison between two sequences generated by the proposed method and modified S-iteration. It can be seen from this figure that both diamond and circle plot tends to zero while the plus sign plot tends to some constant. It was interpreted that both modified S-iteration and proposed methods are converged faster than Ishikawa iteration methods. The proposed method has the same convergence rate as compared with modified S-iteration method. However, the presented method still converged faster than modified S-iteration since the ratio of $|x_n - 1|$ and $|y_n - 1|$ in each iteration step is always less than one (see Definition 3.2.8).

Table 1: Comparative sequences generated by Ishikawa iteration, modified S-iteration and the proposed iteration for numerical experiment of Example 3.2.11

	Comparative sequences			**Rate of convergence between two generated sequences**		
	Proposed iteration	Modified S-iteration	Ishikawa iteration			
n	x_n	y_n	z_n	$\frac{\lvert x_n - 1\rvert}{\lvert z_n - 1\rvert}$	$\frac{\lvert y_n - 1\rvert}{\lvert z_n - 1\rvert}$	$\frac{\lvert x_n - 1\rvert}{\lvert y_n - 1\rvert}$
1	1.1982	1.2157	1.2994	$6.622e-01$	$7.204e-01$	$9.192e-01$
2	1.1151	1.1335	1.2604	$4.419e-01$	$5.126e-01$	$8.621e-01$
3	1.0677	1.0829	1.2282	$2.967e-01$	$3.633e-01$	$8.166e-01$
4	1.0401	1.0516	1.2008	$2.000e-01$	$2.571e-01$	$7.779e-01$
5	1.0239	1.0322	1.1773	$1.352e-01$	$1.818e-01$	$7.438e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	1.0019	1.0031	1.0975	$1.949e-02$	$3.193e-02$	$6.102e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	1.0000	1.0000	1.0310	$4.151e-04$	$9.615e-04$	$4.317e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	1.0000	1.0000	1.0101	$8.919e-06$	$2.852e-05$	$3.126e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
40	1.0000	1.0000	1.0033	$1.925e-07$	$8.418e-07$	$2.287e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	1.0000	1.0000	1.0011	$4.173e-09$	$2.478e-08$	$1.683e-01$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	1.0000	1.0000	1.0003	$9.084e-11$	$7.291e-10$	$1.245e-01$

Table 1 also shows the numerical experiment for supporting our main results and comparing rate of convergence of the proposed method with Ishikawa iteration and modified S-iteration.

CHAPTER IV

CONCLUSION

4.1 Conclusion

The following results are all main theorems of this thesis:

Theorem 4.1.1. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two uniformly $L-$Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1,\infty)$ $(i = 1,2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty}\delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0,1)$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.17) converge strongly to a common fixed point of T_1 and T_2.*

Theorem 4.1.2. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself asymptotically nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\} \subset [1,\infty)$ $(i = 1,2)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0,1)$ such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.17) converge strongly to a common fixed point of T_1 and T_2.*

Theorem 4.1.3. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let*

$T_1, T_2 : C \to X$ be two uniformly $L-$Lipschitzian, nonself generalized asymptotically quasi-nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.16) converge strongly to a common fixed point of T_1 and T_2.

Theorem 4.1.4. *Let X be a real uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself asymptotically nonexpansive mappings of C satisfying condition A' with sequences $\{k_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by the iterative scheme (1.16) converge strongly to a common fixed point of T_1 and T_2.*

Theorem 4.1.5. *Let X be a real Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2 : C \to X$ be two nonself generalized asymptotically quasi-nonexpansive mappings of C with sequences $\{k_n^{(i)}\}$, $\{\delta_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, respectively such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$, $\sum_{n=1}^{\infty} \delta_n^{(i)} < \infty$ and $F = F(T_1) \cap F(T_2) \neq \emptyset$ is closed. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$, $\{\gamma_n\}$, $\{\lambda_n\} \subset [0, 1)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. Then the sequence $\{x_n\}$ defined by the iterative scheme (1.17) converges strongly to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{y \in F} \|x_n - y\|$, $n \geq 1$.*

Theorem 4.1.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C has Property WG. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real* *sequences in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$*. If* $(x_0, z_0), (z_0, x_0) \in E(G)$ *for arbitrary* $x_0 \in C$ *and* $z_0 \in F$*, then* $\{x_n\}$ *converges weakly to a common fixed point of* T_1 *and* T_2*.*

Theorem 4.1.7. *Let* X *be a uniformly convex Banach space. Suppose that* C *has Property* WG*,* $\{\alpha_n\}, \{\beta_n\}$ *are real sequences in* $[\delta, 1-\delta]$ *for some* $\delta \in (0,1)$*,* F *is dominated by* x_0 *and* F *dominates* x_0*. If* $(x_0, z_0), (z_0, x_0) \in E(G)$ *for arbitrary* $x_0 \in C$ *and* $z_0 \in F$*, then* $\{x_n\}$ *converges weakly to a common fixed point of* T_1 *and* T_2*.*

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Publications

Articles

1. **Wongyai K.**, Thianwan T. (2019) Projection type Ishikawa iteration with perturbations for common fixed points of two nonself generalized asymptotically quasi-nonexpansive mappings. Thai. J. Math., 17 (3): 843–859.

Conference presentations

1. **Wongyai K.** (May 15 - 17, 2019). Fixed point algorithms for nonself generalized asymptotically quasi-nonexpansive mappings. The 24th Annual Meeting in Mathematics (AMM 2019), Faculty of Science, Burapha University, Chonburi, Thailand.