FUZZY SOFT SETS OVER FULLY UP-SEMIGROUPS

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บทคัดย่อ

ในงานวิจัยนี้ เราได้นำเสนอเซตย่อยและเซตวิภัชนัยชนิดต่าง ๆ ในกึ่งกรุปยูพีเต็มรูปแบบ เพื่อพิสูจน์ผลลัพธ์ของเซตวิภัชนัยภายใต้การดำเนินการอินเตอร์เซกชันและยูเนียน และยังพิจารณาความสัมพันธ์ระหว่างเซตวิภัชนัยลักษณะเฉพาะที พีชคณิตย่อยยูพีเอส พีชคณิตย่อยยูพีไอ ตัวกรองยูพีเอสใกล้ ตัวกรองยูพีไอใกล้ ตัวกรองยูพีเอส ตัวกรองยูพีไอ ไอดีลยูพีเอส ไอดีลยูพีไอ ไอดีลยูพีเอสอย่างเข้ม และไอดีลยูพีไออย่างเข้ม นอกจากนี้ เรายังได้นำเสนอเซตอ่อนวิภัชนัยเหนือกึ่งกรุปยูพีเต็มรูปแบบอีก 10 ชนิด และพิสูจน์สมบัติของเซตอ่อนวิภัชนัยภายใต้การดำเนินการอินเตอร์เซกชัน(ขยาย) และยูเนียน(จำกัด) และยังพิจารณาความสัมพันธ์ระหว่างบางเงื่อนไขของเซตอ่อนวิภัชนัย พีชคณิตย่อยยูพีเอสอ่อนวิภัชนัย พีชคณิตย่อยยูพีไออ่อนวิภัชนัย ตัวกรองยูพีเอสใกล้อ่อนวิภัชนัย ตัวกรองยูพีไอใกล้อ่อนวิภัชนัย ตัวกรองยูพีเอสอ่อนวิภัชนัย ตัวกรองยูพีไออ่อนวิภัชนัย ไอดีลยูพีเอสอ่อนวิภัชนัย ไอดีลยูพีไออ่อนวิภัชนัย ไอดีลยูพีเอสอย่างเข้มอ่อนวิภัชนัยและไอดีลยูพีไออย่างเข้มอ่อนวิภัชนัย และสุดท้ายเราได้ใช้กฎการแจกแจงของเซตวิภัชนัยใด ๆ เพื่อประยุกต์ไปใช้ในเซตอ่อนวิภัชนัยใด ๆ และพิสูจน์สมบัติของบางการดำเนินการสำหรับเซตอ่อนวิภัชนัยใด ๆ และศึกษาความเกี่ยวข้องของการดำเนินการต่าง ๆ ได้แก่ ยูเนียน(จำกัด) อินเตอร์เซกชัน(ขยาย) แอนด์ และออร์

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ABSTRACT

In this research, we introduce several types of subsets and of fuzzy sets of fully UP-semigroups, investigate the algebraic properties of fuzzy sets under the operations of intersection and union, and discuss the relation between *t*-characteristic fuzzy sets and UP_s-subalgebras (resp., UP_i-subalgebras, near UP_s-filters, near UP_i-filters, UP_s-filters, UP_i-filters, UP_s-ideals, UP_i-ideals, strongly UP_s-ideals and strongly UP_i-ideals). We introduce ten types of fuzzy soft sets over fully UP-semigroups, investigate the algebraic properties of fuzzy soft sets under the operations of (extended) intersection and (restricted) union, and discuss the relation between some conditions of fuzzy soft sets and fuzzy soft UP_s-subalgebras (resp., fuzzy soft UP_i-subalgebras, fuzzy soft near UP_s-filters, fuzzy soft near UP_i-filters, fuzzy soft UP_s-filters, fuzzy soft UP_i-filters, fuzzy soft UP_s-ideals, fuzzy soft UP_i-ideals, fuzzy soft strongly UP_s-ideals, fuzzy soft strongly UP_i-ideals). We apply distributivity laws of several fuzzy sets for any fuzzy sets and study distributivity laws with any fuzzy soft sets. We investigate properties of some operations for fuzzy soft sets and their interrelation with respect to different operations such as "(restricted) union", "(extended) intersection", "AND", and "OR".

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CHAPTER I

INTRODUCTION

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these algebras are BCK-algebras [11], BCI-algebras [12], B-algebras [26], UP-algebras [8] and so on. They are strongly connected with some logic. For example, BCI-algebras introduced by Iséki [12] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [11, 12] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Several researches introduced a new class of algebras related to logical algebras and semigroups such as: In 1993, Jun et al. [15] introduced the notion of BCI-semigroups. In 1998, Jun et al. [20] renamed the BCI-semigroup as the IS-algebra. In 2006, Kim [21] introduced the notion of KS-semigroups. In 2015, Endam and Vilela [6] introduced the notion of JB-semigroups. In 2018, Iampan [9] introduced the notion of fully UP-semigroups.

A fuzzy subset F of a set X is a function from X to a closed interval [0,1]. The concept of a fuzzy subset of a set was first considered by Zadeh [36] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [36], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 1998, Jun et al. [14] applied the notion of fuzzy sets to BCI-semigroups (it was renamed as an IS-algebra for the convenience of study), and introduced the concept of fuzzy I-ideals. In 2000, Roh et al. [30] considered the fuzzification of an associative I-ideal of an IS-algebra. They proved that every fuzzy associative I-ideal is a fuzzy I-ideal. By giving an appropriate example, they verified that a fuzzy I-ideal may not be a fuzzy associative I-ideal. They gave a condition for a fuzzy I-ideal to be a fuzzy associative I-ideal, and they investigated some related properties. In 2003, Jun and Kondo [17] proved that some concepts of BCK/BCI-algebras expressed by a certain formula can be naturally extended to the fuzzy setting and that many results are obtained immediately with the use of our method. Moreover, they proved that these results can be extended to fuzzy IS-algebras. In 2003, Jianming and Dajing [13] introduced the concept of intuitionistic fuzzy associative I-ideals of IS-algebras and they investigated some related properties. In 2007, Prince Williams and Husain [35] studied fuzzy KS-semigroups. In 2016, Endam and Manahon [5] introduced the notion of fuzzy JB-semigroups and they investigated some of its properties.

In 1999, to solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [25]. In 2001, Maji et al. [24] introduced the concept of fuzzy soft sets as a generalization of the standard soft sets, and presented an application of fuzzy soft sets in a decision making problem. In 2010, Jun et al. [18] applied fuzzy soft set for dealing with several kinds of theories in BCK/BCI-algebras. The notions of fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals and fuzzy soft p-ideals are introduced, and related properties are investigated. In 2013, Rehman et al. [28] studied some operations of fuzzy soft sets and give fundamental properties of fuzzy soft sets. They discuss properties of fuzzy soft sets and their interrelation with respect to different operations such as union, intersection, restricted union and extended intersection. Then, they illustrate properties of OR, AND operations by giving counter examples. Also we prove that certain De Morgan's laws hold in fuzzy soft set theory with respect to different operations on fuzzy soft sets.

CHAPTER II

REVIEW OF RELATED LITERATURE AND RESEARCH

Two important classes of logical algebras, BCK and BCI-algebras were introduced by Imai and Iséki [11, 12].

Definition 2.0.1 An algebra $A = (A, \cdot, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(BCI-1) $(\forall x, y, z \in A)(((x \cdot y) \cdot (x \cdot z) \cdot (y \cdot z)) = 0)$,

(BCI-2) $(\forall x, y \in A)((x \cdot (x \cdot y)) \cdot y = x)$,

(BCI-3) $(\forall x \in A)(x \cdot x = 0)$, and

(BCI-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

A BCI-algebra A is called a *BCK-algebra* if it satisfies the following identity:

(BCK) $(\forall x \in A)(0 \cdot x = 0)$.

In 2002, Neggers and Kim [26] introduced the notion of B-algebras.

Definition 2.0.2 An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *B-algebra* if it satisfies the following axioms:

(B-1) $(\forall x \in A)(x \cdot x = 0)$,

(B-2) $(\forall x \in A)(x \cdot 0 = x)$, and

(B-3) $(\forall x, y, z \in A)((x \cdot y) \cdot z = x \cdot (z \cdot (0 \cdot y)))$.

In 2017, Iampan [8] introduced the notion of UP-algebras.

Definition 2.0.3 An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

In 1993, Jun et al. [15] introduced the notion of BCI-semigroups (it was renamed as IS-algebras for the convenience of study).

Definition 2.0.4 An IS-algebra is a nonempty set A together with two binary operations $\cdot$ and $*$ and a constant 0 satisfying the following:

(IS-1) $(A, \cdot, 0)$ is a BCI-algebra,

(IS-2) $(A, *)$ is a semigroup, and

(IS-3) The operation $*$ is left and right distributive over the operation $\cdot$.

In 2006, Kim [21] introduced the notion of KS-semigroups.

Definition 2.0.5 A KS-semigroup is a nonempty set A together with two binary operations $\cdot$ and $*$ and a constant 0 satisfying the following:

(KS-1) $(A, \cdot, 0)$ is a BCK-algebra,

(KS-2) $(A, *)$ is a semigroup, and

(KS-3) The operation $*$ is left and right distributive over the operation $\cdot$.

In 2015, Endam and Vilela [6] introduced the notion of JB-semigroups.

Definition 2.0.6 A JB-semigroup is a nonempty set A together with two binary operations $\cdot$ and $*$ and a constant 0 satisfying the following:

(JB-1) $(A, \cdot, 0)$ is a B-algebra,

(JB-2) $(A, *)$ is a semigroup, and

(JB-3) The operation $*$ is left and right distributive over the operation $\cdot$.

In 2018, Iampan [9] introduced the notion of fully UP-semigroups (in short, f-UP-semigroups).

Definition 2.0.7 An f-UP-semigroup is a nonempty set A together with two binary operations $\cdot$ and $*$ and a constant 0 satisfying the following:

(fUP-1) $(A, \cdot, 0)$ is a UP-algebra,

(fUP-2) $(A, *)$ is a semigroup, and

(fUP-3) The operation $*$ is left and right distributive over the operation $\cdot$.

In 1965, Zadeh [36] introduced the concept of a fuzzy set for the first time.

Definition 2.0.8 A *fuzzy set* F in a nonempty set U (or a *fuzzy subset* of U) is described by its membership function f_{F}. To every point $x \in U$, this function associates a real number $\mathrm{f}_{\mathrm{F}}(x)$ in the interval $[0, 1]$. The number $\mathrm{f}_{\mathrm{F}}(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F, that is,

$F := \{(u, f_F(u)) \mid u \in U\}$. If $A \subseteq U$ and $t \in (0, 1]$, the *t-characteristic function* [19] χ_A^t of U is a function of U into $\{0, t\}$ defined as follows:

$$\chi_A^t(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of t-characteristic function, χ_A^t is a function of U into $\{0, t\} \subset [0, 1]$. We denote the fuzzy set F_A^t in U is described by its membership function χ_A^t, is called the *t-characteristic fuzzy set* of A in U. We say that a fuzzy set F in U is *constant* if its membership function f_F is constant.

In 1999 - 2004, Jun et al. [29, 16] and Jianming and Dajing [13] applied the notion of fuzzy sets to IS-algebras.

Definition 2.0.9 A fuzzy set F in a semigroup $(A, *)$ is called a *fuzzy stable* if $(\forall x, y \in A)(f_F(x * y) \geq f_F(y))$.

Definition 2.0.10 A fuzzy set F in a BCI-algebra $(A, \cdot, 0)$ is called a *fuzzy subalgebra* if $(\forall x, y \in A)(f_F(x \cdot y) \geq \min\{f_F(x), f_F(y)\})$.

Definition 2.0.11 A fuzzy set F in a BCI-algebra $(A, \cdot, 0)$ is called a *fuzzy ideal* of A if it satisfies the following conditions:

(1) $(\forall x \in A)(f_F(0) \geq f_F(x))$, and

(2) $(\forall x, y \in A)(f_F(x) \geq \min\{f_F(x \cdot y), f_F(y)\})$.

Definition 2.0.12 A fuzzy set F in an IS-algebra $(A, \cdot, *, 0)$ is called a *fuzzy I-ideal* of A if it satisfies the following conditions:

(1) F is a fuzzy stable, and

(2) F is a fuzzy ideal of a BCI-algebra A.

Definition 2.0.13 A fuzzy set F in an IS-algebra $(A, \cdot, *, 0)$ is called a *fuzzy associative I-ideal* of A if it satisfies the following conditions:

(1) F is a fuzzy stable, and

(2) $(\forall x, y, z \in A)(\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((x \cdot y) \cdot z), \mathrm{f_F}(y \cdot z)\})$.

In 2016, Endam and Manahon [5] applied the notion of fuzzy sets to JB-semigroups.

Definition 2.0.14 A fuzzy JB-semigroup F of a JB-semigroup $(A, \cdot, *, 0)$ is called a *fuzzy sub JB-semigroup* of A if it satisfies the following conditions:

(1) $(\forall x, y \in A)(\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\})$, and

(2) $(\forall x, y \in A)(\mathrm{f_F}(x * y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\})$.

Definition 2.0.15 A fuzzy JB-semigroup F of a JB-semigroup $(A, \cdot, *, 0)$ is called a *fuzzy JB-ideal* of A if it satisfies the following conditions:

(1) $(\forall x, y, a, b \in A)(\mathrm{f_F}((x \cdot a) \cdot (y \cdot b)) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(a \cdot b)\})$, and

(2) $(\forall x, y \in A)(\mathrm{f_F}(x * y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\})$.

Definition 2.0.16 A fuzzy JB-semigroup F of a JB-semigroup $(A, \cdot, *, 0)$ is called a *fuzzy* JB_s*-ideal* of A if it satisfies the following conditions:

(1) $(\forall x, y, a, b \in A)(\mathrm{f_F}((x \cdot a) \cdot (y \cdot b)) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(a \cdot b)\})$, and

(2) $(\forall x, y \in A)(\mathrm{f_F}(x * y) \geq \max\{\mathrm{f_F}(x), \mathrm{f_F}(y)\})$.

In 2001, Maji et al. [24] introduced the notion of fuzzy soft sets, as a generalization of the standard soft sets.

Definition 2.0.17 Let U be an initial universe set and P be a set of parameters. Let $\mathrm{F}(U)$ denote the set of all fuzzy sets in U. Then $(\widetilde{\mathrm{F}}, E)$ is called a *fuzzy soft set* over U where $E \subseteq P$ and $\widetilde{\mathrm{F}}$ is a mapping given by $\widetilde{\mathrm{F}} \colon E \to \mathrm{F}(U)$.

In general, for every $e \in E$, a fuzzy set,

$$\widetilde{\mathrm{F}}[e] := \{(u, \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(u)) \mid u \in U)\}$$

in U is called *fuzzy value set* of parameter e.

In 2010, Jun et al. [18] applied the notion of fuzzy soft sets to BCK/BCI-algebras.

Definition 2.0.18 Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over a BCK/BCI-algebra $(A, \cdot, 0)$ where E is a subset of P. If there exists $e \in E$ such that $\widetilde{\mathrm{F}}[e]$ is a fuzzy BCK/BCI-algebra in A, we say that $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft BCK/BCI-algebra based on a parameter e over A. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft BCK/BCI-algebra based on a parameter e over A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft BCK/BCI-algebra over A.

Definition 2.0.19 Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over a BCK/BCI-algebra $(A, \cdot, 0)$ where E is a subset of P. If there exists $e \in E$ such that $\widetilde{\mathrm{F}}[e]$ is a fuzzy ideal of A, we say that $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft ideal of A based on a parameter e. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft ideal of A based on all parameters, we say that $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft ideal of A.

CHAPTER III

PRELIMINARIES

Before we begin our study, we will introduce a UP-algebra. From [8], we know that the notion of UP-algebras is a generalization of KU-algebras (see [27]).

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Example 3.0.20 [33] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω.

Example 3.0.21 [33] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω.

In particular, we have $(\mathcal{P}(X), \cdot, \emptyset)$ is the power UP-algebra of type 1 and $(\mathcal{P}(X), *, X)$ is the power UP-algebra of type 2.

Example 3.0.22 [4] Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by,

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right),$$

and

$$(\forall x, y \in \mathbb{N})\left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases}\right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

For more examples of UP-algebras, see [3, 9, 32, 33].

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [8, 9]).

$$(\forall x \in A)(x \cdot x = 0), \tag{3.0.1}$$

$$(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{3.0.2}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{3.0.3}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{3.0.4}$$

$$(\forall x, y \in A)(x \cdot (y \cdot x) = 0), \tag{3.0.5}$$

$$(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{3.0.6}$$

$$(\forall x, y \in A)(x \cdot (y \cdot y) = 0), \tag{3.0.7}$$

$$(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{3.0.8}$$

$$(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{3.0.9}$$

$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{3.0.10}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{3.0.11}$$

$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \text{ and} \tag{3.0.12}$$

$$(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{3.0.13}$$

Definition 3.0.23 [8, 34, 7, 10] A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$,

(2) a *near UP-filter* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y \in A)(y \in S \Rightarrow x \cdot y \in S)$,

(3) a *UP-filter* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$,

(4) a *UP-ideal* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$,

(5) a *strongly UP-ideal* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

We know that the notion of UP-subalgebras is a generalization of near UP-filters, the notion of near UP-filters is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

Definition 3.0.24 A nonempty subset S of a semigroup $(A, *)$ is called

(1) a *subsemigroup* of A if $(\forall x, y \in S)(x * y \in S)$, and

(2) an *ideal* of A if $(\forall x \in A, \forall s \in S)(x * s, s * x \in S)$.

Clearly, an ideal is a subsemigroup.

Lemma 3.0.25 *Let S be a nonempty subset of a UP-algebra $(A, \cdot, 0)$ and $t \in (0, 1]$. Then the constant 0 of A is in S if and only if* $(\forall x \in A)(\chi_S^t(0) \geq \chi_S^t(x))$.

Proof. Assume that $0 \in S$. Then for all $x \in A$, $\chi_S^t(0) = t \geq \chi_S^t(x)$.

Conversely, assume that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Since S is a nonempty subset of A, we have an element a in S, that is, $\chi_S^t(a) = t$. Thus $t \geq \chi_S^t(0) \geq \chi_S^t(a) = t$. So $\chi_S^t(0) = t$, that is, $0 \in S$. □

Definition 3.0.26 ([34, 7]) A fuzzy set F in a UP-algebra $A = (A, \cdot, 0)$ is called

(1) a *fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\})$,

(2) a *fuzzy UP-filter* of A if

  (i) $(\forall x \in A)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x))$, and

  (ii) $(\forall x, y \in A)(\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\})$,

(3) a *fuzzy UP-ideal* of A if

  (i) $(\forall x \in A)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x))$, and

  (ii) $(\forall x, y, z \in A)(\mathrm{f_F}(x \cdot z) \geq \min\{\mathrm{f_F}(x \cdot (y \cdot z)), \mathrm{f_F}(y)\})$,

(4) a *fuzzy strongly UP-ideal* of A if

  (i) $(\forall x \in A)\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and

  (ii) $(\forall x, y, z \in A)(\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((z \cdot y) \cdot (z \cdot x)), \mathrm{f_F}(y)\})$.

Now, we introduce the notion of fuzzy near UP-filters of UP-algebras as follows:

Definition 3.0.27 A fuzzy set F in a UP-algebra $A = (A, \cdot, 0)$ is called a *fuzzy near UP-filter* of A if

(i) $(\forall x \in A)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x))$, and

(ii) $(\forall x, y \in A)(\mathrm{f_F}(x \cdot y) \geq \mathrm{f_F}(y))$.

We know that the notion of fuzzy UP-subalgebras is a generalization of fuzzy near UP-filters, the notion of fuzzy near UP-filters is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals. Moreover, fuzzy strongly UP-ideals and constant fuzzy sets coincide in UP-algebras.

Theorem 3.0.28 *[7] Fuzzy strongly UP-ideals and constant fuzzy sets coincide in UP-algebras.*

Theorem 3.0.29 *Let S be a nonempty subset of a UP-algebra $A = (A, \cdot, 0)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a UP-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-subalgebra of A,

(2) S is a near UP-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy near UP-filter of A,

(3) S is a UP-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-filter of A,

(4) S is a UP-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP-ideal of A, and

(5) S is a strongly UP-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly UP-ideal of A.

Proof. (1) Assume that S is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\chi_S^t(x \cdot y) = t$. Therefore, $\chi_S^t(x \cdot y) = t \geq t = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S^t(x) = 0$ or $\chi_S^t(y) = 0$, so

$$\min\{\chi_S^t(x), \chi_S^t(y)\} = 0.$$

Therefore, $\chi_S^t(x \cdot y) \geq 0 = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy UP-subalgebra of A.

Conversely, assume that F_S^t is a fuzzy UP-subalgebra of A. Let $x, y \in S$. Then $\chi_S^t(y) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy UP-subalgebra of A, we have $t \geq \chi_S^t(x \cdot y) \geq \min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x \cdot y) = t$, that is, $x \cdot y \in S$. Hence, S is a UP-subalgebra of A.

(2) Assume that S is a near UP-filter of A. Since $0 \in S$, it follows from Lemma 3.0.25 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $y \in S$. Then $\chi_S^t(y) = t$. Since S is a near UP-filter of A, we have $x \cdot y \in S$ and so $\chi_S^t(x \cdot y) = t$. Therefore, $\chi_S^t(x \cdot y) = t \geq t = \chi_S^t(y)$.

Case 2: $y \notin S$. Then $\chi_S^t(y) = 0$. Thus $\chi_S^t(x \cdot y) \geq 0 = \chi_S^t(y)$.

Hence, F_S^t is a fuzzy near UP-filter of A.

Conversely, assume that F_S^t is a fuzzy near UP-filter of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 3.0.25 that $0 \in S$. Next, let $x, y \in A$ be such that $y \in S$. Then $\chi_S^t(y) = t$. Since F_S^t is a fuzzy near UP-filter of A, we have $t \geq \chi_S^t(x \cdot y) \geq \chi_S^t(y) = t$. Thus $\chi_S^t(x \cdot y) = t$, that is, $x \cdot y \in S$. Hence, S is a near UP-filter of A.

(3) Assume that S is a UP-filter of A. Since $0 \in S$, it follows from Lemma

3.0.25 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$. Thus $\chi_S^t(y) = t \geq \chi_S^t(x \cdot y) = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x \notin S$, then $\chi_S^t(x) = 0$. Thus $\chi_S^t(y) \geq 0 = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$. If $y \notin S$, then $\chi_S^t(y) = 0$. Since S is a UP-filter of A, we have $x \cdot y \notin S$ or $x \notin S$ and so $\chi_S^t(x \cdot y) = 0$ or $\chi_S^t(x) = 0$. Thus $\chi_S^t(y) = 0 \geq 0 = \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\}$.

Hence, F_S^t is a fuzzy UP-filter of A.

Conversely, assume that F_S^t is a fuzzy UP-filter of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 3.0.25 that $0 \in S$. Next, let $x, y \in A$ be such that $x \cdot y \in S$ and $x \in S$. Then $\chi_S^t(x \cdot y) = t = \chi_S^t(x)$, so $\min\{\chi_S^t(x \cdot y), \chi_S^t(x)\} = t$. Since F_S^t is a fuzzy UP-filter of A, we have $t \geq \chi_S^t(y) \geq \min\{\chi_S^t(x \cdot y), \chi_S^t(x)\} = t$. Thus $\chi_S^t(y) = t$, that is, $y \in S$. Hence, S is a UP-filter of A.

(4) Assume that S is a UP-ideal of A. Since $0 \in S$, it follows from Lemma 3.0.25 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z), y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Since S is a UP-ideal of A, we have $x \cdot z \in S$ and so $\chi_S^t(x \cdot z) = t$. Thus $\chi_S^t(x \cdot z) = t \geq t = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Case 2: $x \cdot (y \cdot z) \notin S$ or $y \notin S$. Then $\chi_S^t(x \cdot (y \cdot z)) = 0$ or $\chi_S^t(y) = 0$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = 0$. Thus $\chi_S^t(x \cdot z) \geq 0 = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy UP-ideal of A.

Conversely, assume that F_S^t is a fuzzy UP-ideal of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 3.0.25 that $0 \in S$. Next, let $x, y, z \in A$ such that $x \cdot (y \cdot z) \in S$ and $y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so

$\min\{\chi^t_S(x \cdot (y \cdot z)), \chi^t_S(y)\} = t$. Since F^t_S is a fuzzy UP-ideal of A, we have $t \geq \chi^t_S(x \cdot z) \geq \min\{\chi^t_S(x \cdot (y \cdot z)), \chi^t_S(y)\} = t$. Thus $\chi^t_S(x \cdot z) = t$, that is, $x \cdot z \in S$. Hence, S is a UP-ideal of A.

(5) It is straightforward by Theorem 3.0.28, and A is the only one strongly UP-ideal of itself. □

Definition 3.0.30 [31] A fuzzy set F in a semigroup $A = (A, *)$ is called

(1) a *fuzzy subsemigroup* of A if

$$(\forall x, y \in A)(\mathrm{f_F}(x * y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}), \text{ and}$$

(2) a *fuzzy ideal* of A if for any $x, y \in A$,

$$(\forall x, y \in A)(\mathrm{f_F}(x * y) \geq \max\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}).$$

Clearly, a fuzzy ideal is a fuzzy subsemigroup.

Theorem 3.0.31 *Let S be a nonempty subset of a semigroup $A = (A, *)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a subsemigroup of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy subsemigroup of A, and

(2) S is an ideal of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy ideal of A.

Proof. (1) Assume that S is a subsemigroup of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi^t_S(x) = t = \chi^t_S(y)$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Since S is a subsemigroup of A, we have $x*y \in S$ and so $\chi^t_S(x*y) = t$. Therefore, $\chi^t_S(x * y) = t \geq t = \min\{\chi^t_S(x), \chi^t_S(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S^t(x) = 0$ or $\chi_S^t(y) = 0$, so

$$\min\{\chi_S^t(x), \chi_S^t(y)\} = 0.$$

Therefore, $\chi_S^t(x * y) \geq 0 = \min\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy subsemigroup of A.

Conversely, assume that F_S^t is a fuzzy subsemigroup of A. Let $x, y \in S$. Then $\chi_S^t(y) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy subsemigroup of A, we have $t \geq \chi_S^t(x * y) \geq \min\{\chi_S^t(x), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x * y) = t$, that is, $x * y \in S$. Hence, S is a subsemigroup of A.

(2) Assume that S is an ideal of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\max\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is an ideal of A, we have $x * y \in S$ and so $\chi_S^t(x * y) = t$. Therefore, $\chi_S^t(x * y) = t \geq t = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x * y \in S$, then $\chi_S^t(x * y) = t$. Therefore, $\chi_S^t(x*y) = t \geq \max\{\chi_S^t(x), \chi_S^t(y)\}$. If $x*y \notin S$, then $x, y \notin S$. Thus $\chi_S^t(x*y) = 0$ and $\chi_S^t(x) = 0 = \chi_S^t(y)$. Therefore, $\chi_S^t(x * y) = 0 \geq 0 = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy ideal of A.

Conversely, assume that F_S^t is a fuzzy ideal of A. Let $s \in S$ and $x \in A$. Then $\chi_S^t(s) = t$, so $\max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Since F_S^t is a fuzzy ideal of A, we have $t \geq \chi_S^t(s*x), \chi_S^t(x*s) \geq \max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Thus $\chi_S^t(s*x) = t = \chi_S^t(x*s)$, that is $s * x, x * s \in S$. Hence, S is an ideal of A. □

Definition 3.0.32 [23] Let $\{\mathrm{F}_i\}_{i \in I}$ be a nonempty family of fuzzy sets in a nonempty set U where I is an arbitrary index set. The *intersection* of F_i, denoted by $\bigcap_{i \in I} \mathrm{F}_i$, is described by its membership function $\mathrm{f}_{\bigcap_{i\in I} \mathrm{F}_i}$ which defined as follows:

$$(\forall x \in U)(\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x) = \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}).$$

The *union* of F_i, denoted by $\bigcup_{i\in I}\mathrm{F}_i$, is described by its membership function $\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}$ which defined as follows:

$$(\forall x \in U)(\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x) = \sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}).$$

Definition 3.0.33 [23] Let F and G be fuzzy sets in a nonempty set U. Then $\mathrm{F} \leq \mathrm{G}$ is defined by $\mathrm{f}_{\mathrm{F}}(x) \leq \mathrm{f}_{\mathrm{G}}(x)$ for all $x \in U$.

Definition 3.0.34 [22] Let F and G be fuzzy sets in a semigroup $A = (A, *)$. Then the *product* of F and G, denoted by $\mathrm{F} \circ \mathrm{G}$, is described by their membership function f_{F} and f_{G}, respectively which defined as follows: For all $x \in A$,

$$(\mathrm{f}_{\mathrm{F}} \circ \mathrm{f}_{\mathrm{G}})(x) = \begin{cases} \sup\{\min\{\mathrm{f}_{\mathrm{F}}(y), \mathrm{f}_{\mathrm{G}}(z)\}\}_{x=y*z} & \text{if } \exists y, z \in A \text{ such that } x = y * z, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.0.35 [22] The semigroup A itself is a fuzzy set of A, denoted by A such that $\mathrm{f}_{\mathrm{A}}(x) = 1$ for all $x \in A$.

Lemma 3.0.36 *[22] Let* F *be a fuzzy set in a semigroup* $A = (A, *)$*. Then*

(1) F *is a fuzzy subsemigroup of* A *if and only if it satisfies the condition*

$$\mathrm{F} \circ \mathrm{F} \leq \mathrm{F}. \tag{3.0.14}$$

(2) F *is a fuzzy ideal of* A *if and only if it satisfies the condition*

$$\mathrm{A} \circ \mathrm{F} \leq \mathrm{F} \text{ and } \mathrm{F} \circ \mathrm{A} \leq \mathrm{F}. \tag{3.0.15}$$

Theorem 3.0.37 *Let* F_i *and* F *be fuzzy sets in a nonempty set* X *where* I *is a nonempty set. Then the following properties hold:*

(1) $\mathrm{F} \cap (\bigcup_{i\in I} \mathrm{F}_i) = \bigcup_{i\in I}(\mathrm{F} \cap \mathrm{F}_i)$,

(2) $(\bigcup_{i\in I} \mathrm{F}_i) \cap \mathrm{F} = \bigcup_{i\in I}(\mathrm{F}_i \cap \mathrm{F})$,

(3) $\mathrm{F} \cup (\bigcap_{i\in I} \mathrm{F}_i) = \bigcap_{i\in I}(\mathrm{F} \cup \mathrm{F}_i)$, *and*

(4) $(\bigcap_{i\in I} \mathrm{F}_i) \cup \mathrm{F} = \bigcap_{i\in I}(\mathrm{F}_i \cup \mathrm{F})$.

Proof. Let $x \in X$. (1) First, we investigate left hand side of the equality. Assume that $\bigcup_{i\in I} \mathrm{F}_i = \mathrm{F}^{\cup}$. Then $\mathrm{F} \cap (\bigcup_{i\in I} \mathrm{F}_i) = \mathrm{F} \cap \mathrm{F}^{\cup}$. Also,

$$\begin{aligned}
\mathrm{f}_{\mathrm{F}\cap\mathrm{F}^{\cup}}(x) &= \min\{\mathrm{f}_{\mathrm{F}(x)}, \mathrm{f}_{\mathrm{F}^{\cup}}(x)\} \\
&= \min\{\mathrm{f}_{\mathrm{F}(x)}, \mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x)\} \\
&= \min\{\mathrm{f}_{\mathrm{F}(x)}, \sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\}.
\end{aligned}$$

Consider the right hand side of the equality. Assume that $\mathrm{F} \cap \mathrm{F}_i = \mathrm{F}_i^{\cap}$ for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i^{\cap}}(x) &= \sup\{\mathrm{f}_{\mathrm{F}_i^{\cap}}(x)\}_{i\in I} \\
&= \sup\{\mathrm{f}_{\mathrm{F}\cap\mathrm{F}_i}(x)\}_{i\in I} \\
&= \sup\{\min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}_i}(x)\}\}_{i\in I}.
\end{aligned}$$

It is clear that $\min\{\mathrm{f}_{\mathrm{F}(x)}, \sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\} = \sup\{\min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}_i}(x)\}\}_{i\in I}$. Therefore, $\mathrm{F} \cap (\bigcup_{i\in I} \mathrm{F}_i) = \bigcup_{i\in I}(\mathrm{F} \cap \mathrm{F}_i)$.

(2) By using techniques as in (1), then (2) can is derived.

(3) First, we investigate left hand side of the equality. Assume that $\bigcap_{i\in I} \mathrm{F}_i = \mathrm{F}^{\cap}$. Then $\mathrm{F} \cup (\bigcap_{i\in I} \mathrm{F}_i) = \mathrm{F} \cup \mathrm{F}^{\cap}$. Also,

$$\begin{aligned}
\mathrm{f}_{\mathrm{F}\cup\mathrm{F}^{\cap}}(x) &= \max\{\mathrm{f}_{\mathrm{F}(x)}, \mathrm{f}_{\mathrm{F}^{\cap}}(x)\} \\
&= \max\{\mathrm{f}_{\mathrm{F}(x)}, \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x)\}
\end{aligned}$$

$$= \max\{f_{F(x)}, \inf\{f_{F i}(x)\}_{i\in I}\}.$$

Consider the right hand side of the equality. Assume that $F \cup F_i = F_i^{\cup}$ for all $i \in I$. Then

$$\begin{aligned} f_{\bigcap_{i\in I} F_i^{\cup}}(x) &= \inf\{f_{F_i^{\cup}}(x)\}_{i\in I} \\ &= \inf\{f_{F\cup F_i}(x)\}_{i\in I} \\ &= \inf\{\max\{f_F(x), f_{F_i}(x)\}\}_{i\in I}. \end{aligned}$$

It is clear that $\max\{f_{F(x)}, \inf\{f_{F i}(x)\}_{i\in I}\} = \inf\{\max\{f_F(x), f_{F_i}(x)\}\}_{i\in I}$. Therefore, $F \cup (\bigcap_{i\in I} F_i) = \bigcap_{i\in I}(F \cup F_i)$.

(4) By using techniques as in (3), then (4) can is derived. $\square$

CHAPTER IV

RESULTS

4.1 Special subsets of fully UP-semigroups

In this section, we introduce the notions of $\mathrm{UP_s}$-subalgebras, $\mathrm{UP_i}$-subalgebras, near $\mathrm{UP_s}$-filters, near $\mathrm{UP_i}$-filters, $\mathrm{UP_s}$-filters, $\mathrm{UP_i}$-filters, $\mathrm{UP_s}$-ideals, $\mathrm{UP_i}$-ideals, strongly $\mathrm{UP_s}$-ideals, and strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples and prove its generalizations.

From now on, we shall let A be an f-UP-semigroup $A = (A, \cdot, *, 0)$ unless otherwise specified.

Definition 4.1.1 A subset S of an f-UP-semigroup A is called

(1) a *UP_s-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$, and

(2) a *UP_i-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 4.1.2, 4.1.13, and 4.1.18 directly from Definition 3.0.24.

Theorem 4.1.2 *Every UP_i-subalgebra of A is a UP_s-subalgebra of A.*

Example 4.1.3 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	3	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a $\mathrm{UP_s}$-subalgebra of A. Since $1 \in S$ and $3 \in A$ but $3 * 1 = 3 \notin S$, we have S is not an ideal of $(A, *)$. Thus S is not a $\mathrm{UP_i}$-subalgebra of A.

Definition 4.1.4 A subset S of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is called

(1) a *near* $\mathit{UP_s}$*-filter* of A if S is a near UP-filter of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$, and

(2) a *near* $\mathit{UP_i}$*-filter* of A if S is a near UP-filter of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 4.1.5, 4.1.7, 4.1.10, 4.1.12, 4.1.15, 4.1.17, 4.1.20, and 4.1.22 directly from a result quoted in Definition 3.0.23.

Theorem 4.1.5 *Every near* $\mathit{UP_s}$*-filter of* A *is a* $\mathit{UP_s}$*-subalgebra of* A*.*

Example 4.1.6 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 2\}$. Then S is a $\mathrm{UP_s}$-subalgebra of A. Since $2 \in S$ but $3 \cdot 2 = 1 \notin S$, we have S is not a near UP-filter of $(A, \cdot, 0)$. Thus S is not a near $\mathrm{UP_s}$-filter of A.

Theorem 4.1.7 *Every near* $\mathit{UP_i}$*-filter of* A *is a* $\mathit{UP_i}$*-subalgebra of* A*.*

In Example 4.1.6, we have S is a $\mathrm{UP_i}$-subalgebra of A. Since S is not a near UP-filter of $(A, \cdot, 0)$, we have S is not a near $\mathrm{UP_i}$-filter of A.

Theorem 4.1.8 *Every near* UP_{i}*-filter of* A *is a near* UP_{s}*-filter of* A*.*

In Example 4.1.3, we have S is a near $\mathrm{UP_s}$-filter of A. Since S is not an ideal of $(A, *)$, we have S is not a near $\mathrm{UP_i}$-filter of A.

Definition 4.1.9 A subset S of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is called

(1) a *UP_{s}-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$, and

(2) a *UP_{i}-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 4.1.10 *Every* UP_{s}*-filter of* A *is a near* UP_{s}*-filter of* A*.*

Example 4.1.11 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 2\}$. Then S is a near $\mathrm{UP_s}$-filter of A. Since $2 \cdot 1 = 0 \in S$ and $2 \in S$ but $1 \notin S$, we have S is not a UP-filter of $(A, \cdot, 0)$. Thus S is not a $\mathrm{UP_s}$-filter of A.

Theorem 4.1.12 *Every* UP_{i}*-filter of* A *is a near* UP_{i}*-filter of* A*.*

In Example 4.1.11, we have S is a near $\mathrm{UP_i}$-filter of A. Since S is not a UP-filter of $(A, \cdot, 0)$, we have S is not a $\mathrm{UP_i}$-filter of A.

Theorem 4.1.13 *Every* UP_{i}*-filter of* A *is a* UP_{s}*-filter of* A*.*

In Example 4.1.3, we have S is a UP$_\mathrm{s}$-filter of A. Since S is not an ideal of $(A, *)$, we have S is not a UP$_\mathrm{i}$-filter of A.

Definition 4.1.14 A subset S of an f-UP-semigroup A is called

(1) a *UP$_\mathrm{s}$-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$, and

(2) a *UP$_\mathrm{i}$-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 4.1.15 *Every UP$_\mathrm{s}$-ideal of A is a UP$_\mathrm{s}$-filter of A.*

Example 4.1.16 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1\}$. Then S is a UP$_\mathrm{s}$-filter of A. Since $2 \cdot (1 \cdot 3) = 0 \in S$ and $1 \in S$ but $2 \cdot 3 = 2 \notin S$, we have S is not a UP-ideal of $(A, \cdot, 0)$. Thus S is not a UP$_\mathrm{s}$-ideal of A.

Theorem 4.1.17 *Every UP$_\mathrm{i}$-ideal of A is a UP$_\mathrm{i}$-filter of A.*

In Example 4.1.16, we have S is a UP$_\mathrm{i}$-filter of A. Since S is not a UP-ideal of $(A, \cdot, 0)$, we have S is not a UP$_\mathrm{i}$-ideal of A.

Theorem 4.1.18 *Every UP$_\mathrm{i}$-ideal of A is a UP$_\mathrm{s}$-ideal of A.*

In Example 4.1.3, we have S is a UP$_\mathrm{s}$-ideal of A. Since S is not an ideal of $(A, *)$, we have S is not a UP$_\mathrm{i}$-ideal of A.

Definition 4.1.19 A subset S of an f-UP-semigroup A is called

(1) a *strongly* UP_{s}*-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$, and

(2) a *strongly* UP_{i}*-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 4.1.20 *Every strongly* UP_{s}*-ideal of* A *is a* UP_{s}*-ideal of* A.

Example 4.1.21 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a $\mathrm{UP_s}$-ideal of A. Since $S \neq A$, we have S is not a strongly UP-ideal of $(A, \cdot, 0)$. Thus S is not a strongly $\mathrm{UP_s}$-ideal of A.

Theorem 4.1.22 *Every strongly* UP_{i}*-ideal of* A *is a* UP_{i}*-ideal of* A.

In Example 4.1.21, we have S is a $\mathrm{UP_i}$-ideal of A. Since S is not a strongly UP-ideal of $(A, \cdot, 0)$, we have S is not a strongly $\mathrm{UP_i}$-ideal of A.

Theorem 4.1.23 *Strongly* UP_{s}*-ideals and strongly* UP_{i}*-ideals coincide in* A *and it is only* A.

Proof. It is straightforward by A is the only one strongly UP-ideal of itself. □

4.2 Fuzzy sets in fully UP-semigroups

In this section, we introduce the notions of fuzzy $\mathrm{UP_s}$-subalgebras, fuzzy $\mathrm{UP_i}$-subalgebras, fuzzy $\mathrm{UP_s}$-filters, fuzzy $\mathrm{UP_i}$-filters, fuzzy $\mathrm{UP_s}$-ideals, fuzzy $\mathrm{UP_i}$-ideals, fuzzy strongly $\mathrm{UP_s}$-ideals, and fuzzy strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples, prove its generalizations and investigate the algebraic properties of fuzzy sets under the operations of intersection and union.

Definition 4.2.1 A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy* $\mathit{UP_s}$*-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$, and

(2) a *fuzzy* $\mathit{UP_i}$*-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy $\mathrm{UP_i}$-subalgebra is a fuzzy $\mathrm{UP_s}$-subalgebra.

In Example 4.1.21, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and } \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-subalgebra of A. Since $\mathrm{f_F}(2 * 3) = \mathrm{f_F}(1) = 0.4 \ngeq 0.5 = \max\{0.5, 0.2\} = \max\{\mathrm{f_F}(2), \mathrm{f_F}(3)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-subalgebra of A.

Theorem 4.2.2 *The intersection of any nonempty family of fuzzy* $\mathit{UP_s}$*-subalgebras of* A *is also a fuzzy* $\mathit{UP_s}$*-subalgebra of* A.

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-subalgebra of A for all $i \in I$. Then

$$\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x \cdot y) = \inf\{\mathrm{f}_{\mathrm{F}_i}(x \cdot y)\}_{i\in I}$$

$$\begin{aligned}
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\} \text{ and}\\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x*y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A. □

In Example 4.1.21, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follows:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}1}$	0.7	0.5	0.7	0.3
$\mathrm{f}_{\mathrm{F}2}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3*2) = \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1) = 0.5 \not\geq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \cup \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Theorem 4.2.3 *A nonempty subset S of A is a UP_{s}-subalgebra of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy UP_{s}-subalgebra of A.*

Proof. It is straightforward by Theorem 3.0.31 (1) and Theorem 3.0.29 (1). □

Theorem 4.2.4 *The intersection of any nonempty family of fuzzy UP_{i}-subalgebras of A is also a fuzzy UP_{i}-subalgebra of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-subalgebra of A for all $i \in I$. Then

$$\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot y) = \inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot y)\}_{i\in I}$$

$$
\begin{aligned}
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\} \text{ and} \\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I} \\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}
$$

Hence, $\bigcap_{i\in I}\mathrm{F}_i$ is a fuzzy UP$_\mathrm{i}$-subalgebra of A. □

In Example 4.1.16, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follows:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.9	0.7	0.1	0.1
$\mathrm{f}_{\mathrm{F}_2}$	0.8	0.4	0.5	0.6

Then F_1 and F_2 are fuzzy UP$_\mathrm{i}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1\cdot 3) = \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(2) = 0.5 \ngeq 0.6 = \min\{0.7, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1), \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3)\}$, we have $\mathrm{F}_1 \cup \mathrm{F}_2$ is not a fuzzy UP$_\mathrm{i}$-subalgebra of A.

Theorem 4.2.5 *A nonempty subset S of A is a UP$_\mathrm{i}$-subalgebra of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy UP$_\mathrm{i}$-subalgebra of A.*

Proof. It is straightforward by Theorem 3.0.31 (2) and Theorem 3.0.29 (1). □

Definition 4.2.6 A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy near UP$_\mathrm{s}$-filter* of A if F is a fuzzy near UP-filter of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$, and

(2) a *fuzzy near UP$_\mathrm{i}$-filter* of A if F is a fuzzy near UP-filter of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy near $\mathrm{UP_i}$-filter is a fuzzy near $\mathrm{UP_s}$-filter.

In Example 4.1.21, we define a membership function f_F as follows:

$$f_F(0) = 1,\ f_F(1) = 0.4,\ f_F(2) = 0.5,\ \text{and } f_F(3) = 0.2.$$

Then F is a fuzzy near $\mathrm{UP_s}$-filter of A. Since $f_F(2 * 3) = f_F(1) = 0.4 \ngeq 0.5 = \max\{0.5, 0.2\} = \max\{f_F(2), f_F(3)\}$, we have F is not a fuzzy near $\mathrm{UP_i}$-fiter of A.

Theorem 4.2.7 *The intersection of any nonempty family of fuzzy near UP_s-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near UP_s-filter.*

Proof. Let F_i be a fuzzy near $\mathrm{UP_s}$-filter of an f-UP-semigroup $A = (A, \cdot, *, 0)$ for all $i \in I$. Then

$$\begin{aligned}
f_{\bigcap_{i\in I} F_i}(0) &= \inf\{f_{F_i}(0)\}_{i\in I} \\
&\geq \inf\{f_{F_i}(x)\}_{i\in I} \\
&= f_{\bigcap_{i\in I} F_i}(x), \\
f_{\bigcap_{i\in I} F_i}(x \cdot y) &= \inf\{f_{F_i}(x \cdot y)\}_{i\in I} \\
&\geq \inf\{f_{F_i}(y)\}_{i\in I} \\
&= f_{\bigcap_{i\in I} F_i}(y), \text{and} \\
f_{\bigcap_{i\in I} F_i}(x * y) &= \inf\{f_{F_i}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{f_{F_i}(x), f_{F_i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{f_{F_i}(x)\}_{i\in I}, \inf\{f_{F_i}(y)\}_{i\in I}\} \\
&= \min\{f_{\bigcap_{i\in I} F_i}(x), f_{\bigcap_{i\in I} F_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I} F_i$ is a fuzzy near $\mathrm{UP_s}$-filter of A. □

In Example 4.1.21, we define two membership functions f_{F_1} and f_{F_2} as follows:

A	0	1	2	3
f_{F_1}	1	0.7	1	0.5
f_{F_2}	1	0.5	0.3	0.8

Then F_1 and F_2 are fuzzy near UP_s-filters of A but $F_1 \cup F_2$ is not a fuzzy near UP_s-filter of A. Indeed, $f_{F_1\cup F_2}(3 * 2) = f_{F_1\cup F_2}(1) = 0.7 \ngeq 0.8 = \min\{0.8, 1\} = \min\{f_{F_1\cup F_2}(3), f_{F_1\cup F_2}(2)\}$.

Theorem 4.2.8 *A nonempty subset S of A is a near UP_s-filter of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy near UP_s-filter of A.*

Proof. It is straightforward by Theorem 3.0.31 (1) and Theorem 3.0.29 (2). □

Theorem 4.2.9 *The intersection of any nonempty family of fuzzy near UP_i-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near UP_i-filter.*

Proof. Let F_i be a fuzzy near UP_i-filter of an f-UP-semigroup $A = (A, \cdot, *, 0)$ for all $i \in I$. Then, by the proof of Theorem 4.2.7, we have $f_{\bigcap_{i\in I} F_i}(0) \geq f_{\bigcap_{i\in I} F_i}(x)$ and $f_{\bigcap_{i\in I} F_i}(x \cdot y) \geq f_{\bigcap_{i\in I} F_i}(y)$. Thus

$$
\begin{aligned}
f_{\bigcap_{i\in I} F_i}(x * y) &= \inf\{f_{F_i}(x * y)\}_{i\in I} \\
&\geq \inf\{\max\{f_{F_i}(x), f_{F_i}(y)\}\}_{i\in I} \\
&\geq \max\{\inf\{f_{F_i}(x)\}_{i\in I}, \inf\{f_{F_i}(y)\}_{i\in I}\} \\
&= \max\{f_{\bigcap_{i\in I} F_i}(x), f_{\bigcap_{i\in I} F_i}(y)\}.
\end{aligned}
$$

Hence, $\bigcap_{i\in I} F_i$ is a fuzzy near UP_i-filter of A. □

Theorem 4.2.10 *The union of any nonempty family of fuzzy near UP_i-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near UP_i-filter.*

Proof. Let F_i be a fuzzy near UP_i-filter of an f-UP-semigroup $A = (A, \cdot, *, 0)$ for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(0) &= \sup\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I} \\
&\geq \sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x\cdot y) &= \sup\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I} \\
&\geq \sup\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I} \\
&= \mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(y), \text{and} \\
\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x * y) &= \sup\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I} \\
&\geq \sup\{\max\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \max\{\sup\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \sup\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcup_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigcup_{i\in I}\mathrm{F}_i$ is a fuzzy near $\mathrm{UP_i}$-filter of A. □

Theorem 4.2.11 *A nonempty subset S of A is a near UP_{i}-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy near UP_{i}-filter of A.*

Proof. It is straightforward by Theorem 3.0.31 (2) and Theorem 3.0.29 (2). □

We have Theorem 4.2.12, 4.2.13, 4.2.20, 4.2.22, 4.2.28, 4.2.29, and 4.2.36 directly from a result quoted in Definition 3.0.26.

Theorem 4.2.12 *Every fuzzy near UP_{s}-filter of an f-UP-semigroup is a fuzzy UP_{s}-subalgebra.*

In Example 4.1.6, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1, \mathrm{f_F}(1) = 0.8, \mathrm{f_F}(2) = 0.9, \text{ and } \mathrm{f_F}(3) = 0.7.$$

Then F is a fuzzy $\mathrm{UP_s}$-subalgebra of A. Since $\mathrm{f_F}(1\cdot 2) = \mathrm{f_F}(1) = 0.8 \ngeq 0.9 = \mathrm{f_F}(2)$, we have F is not a fuzzy near $\mathrm{UP_s}$-filter of A.

Theorem 4.2.13 *Every fuzzy near UP_{i}-filter of an f-UP-semigroup is a fuzzy UP_{i}-subalgebra.*

In Example 4.1.6, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 0.8, \mathrm{f_F}(1) = 0.4, \mathrm{f_F}(2) = 0.8, \text{ and } \mathrm{f_F}(3) = 0.3.$$

Then F is a fuzzy $\mathrm{UP_i}$-subalgebra of A. Since $\mathrm{f_F}(1\cdot 2) = \mathrm{f_F}(1) = 0.4 \ngeq 0.8 = \mathrm{f_F}(2)$, we have F is not a fuzzy near $\mathrm{UP_i}$-filter of A.

Definition 4.2.14 A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy UP_{s}-filter* of A if F is a fuzzy UP-filter of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$, and

(2) a *fuzzy UP_{i}-filter* of A if F is a fuzzy UP-filter of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy $\mathrm{UP_i}$-filter is a fuzzy $\mathrm{UP_s}$-filter.

In Example 4.1.21, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and } \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-filter of A. Since$\mathrm{f_F}(2 * 3) = \mathrm{f_F}(1) = 0.4 \ngeq 0.5 = \max\{0.5, 0.2\} = \max\{\mathrm{f_F}(2), \mathrm{f_F}(3)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-filter of A.

Theorem 4.2.15 *The intersection of any nonempty family of fuzzy UP_{s}-filters of A is also a fuzzy UP_{s}-filter of A.*

Proof. Let F_i be a fuzzy UP_s-filter of A for all $i \in I$. Then

$$\begin{aligned}
f_{\bigcap_{i\in I} F_i}(0) &= \inf\{f_{F_i}(0)\}_{i\in I} \\
&\geq \inf\{f_{F_i}(x)\}_{i\in I} \\
&= f_{\bigcap_{i\in I} F_i}(x), \\
f_{\bigcap_{i\in I} F_i}(y) &= \inf\{f_{F_i}(y)\}_{i\in I} \\
&\geq \inf\{\min\{f_{F_i}(x \cdot y), f_{F_i}(x)\}\}_{i\in I} \\
&= \min\{\inf\{f_{F_i}(x \cdot y)\}_{i\in I}, \inf\{f_{F_i}(x)\}_{i\in I}\} \\
&= \min\{f_{\bigcap_{i\in I} F_i}(x \cdot y), f_{\bigcap_{i\in I} F_i}(x)\}, \text{ and} \\
f_{\bigcap_{i\in I} F_i}(x * y) &= \inf\{f_{F_i}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{f_{F_i}(x), f_{F_i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{f_{F_i}(x)\}_{i\in I}, \inf\{f_{F_i}(y)\}_{i\in I}\} \\
&= \min\{f_{\bigcap_{i\in I} F_i}(x), f_{\bigcap_{i\in I} F_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I} F_i$ is a fuzzy UP_s-filter of A. □

In Example 4.1.21, we define two membership functions f_{F1} and f_{F2} as follows:

A	0	1	2	3
f_{F_1}	0.7	0.5	0.7	0.3
f_{F_2}	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy UP_s-filters of A. Since $f_{F_1\cup F_2}(2*3) = f_{F_1\cup F_2}(1) = 0.5 \ngeq 0.6 = \min\{0.7, 0.6\} = \min\{f_{F_1\cup F_2}(2), f_{F_1\cup F_2}(3)\}$, we have $F_1 \cup F_2$ is not a fuzzy UP_s-filter of A.

Theorem 4.2.16 *A nonempty subset S of A is a UP_s-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_s-filter of A.*

Proof. It is straightforward by Theorem 3.0.31 (1) and Theorem 3.0.29 (3). □

Theorem 4.2.17 *The intersection of any nonempty family of fuzzy UP*$_{\rm i}$*-filters of* A *is also a fuzzy UP*$_{\rm i}$*-filter of* A.

Proof. Let F_i be a fuzzy UP$_{\rm i}$-filter of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\mathrm{F}_i}(x)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot y)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x)\}, \text{ and} \\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x*y)\}_{i\in I} \\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{F}_i$ is a fuzzy UP$_{\rm i}$-filter of A. □

Example 4.2.18 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. We define two membership functions

f_{F1} and f_{F2} as follows:

A	0	1	2	3
f_{F_1}	0.9	0.9	0.5	0.5
f_{F_2}	1	0.5	0.6	0.5

Then F_1 and F_2 are fuzzy UP_i-filters of A. Since $f_{F_1 \cup F_2}(3) = 0.5 \ngeq 0.6 = \min\{0.9, 0.6\} = \min\{f_{F_1 \cup F_2}(1), f_{F_1 \cup F_2}(2)\} = \min\{f_{F_1 \cup F_2}(2 \cdot 3), f_{F_1 \cup F_2}(2)\}$, we have $F_1 \cup F_2$ is not a fuzzy UP_i-filter of A.

Theorem 4.2.19 *A nonempty subset S of A is a UP_i-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_i-filter of A.*

Proof. It is straightforward by Theorem 3.0.31 (2) and Theorem 3.0.29 (3). □

Theorem 4.2.20 *Every fuzzy UP_s-filter of an f-UP-semigroup is a fuzzy near UP_s-filter.*

The following example shows that the converse of Theorem 4.2.20 is not true.

Example 4.2.21 Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	2

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. We define a membership function f_F as follows:

$$f_F(0) = 1, f_F(1) = 0.7, f_F(2) = 0.9, \text{ and } f_F(3) = 0.8.$$

Then F is a fuzzy near UP_s-filter of A. Since $\text{f}_\text{F}(1) = 0.7 \ngeq 0.8 = \min\{1, 0.8\} = \min\{\text{f}_\text{F}(0), \text{f}_\text{F}(3)\} = \min\{\text{f}_\text{F}(3 \cdot 1), \text{f}_\text{F}(3)\}$, we have F is not a fuzzy UP_s-filter of A.

Theorem 4.2.22 *Every fuzzy* UP_i*-filter of an* f*-UP-semigroup is a fuzzy near* UP_i*-filter.*

In Example 4.2.21, we have F is a fuzzy near UP_i-filter of A but it is not a fuzzy UP_i-filter of A.

Definition 4.2.23 A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy* UP_s*-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$, and

(2) a *fuzzy* UP_i*-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy UP_i-ideal is a fuzzy UP_s-ideal.

In Example 4.1.21, we define a membership function f_F as follows:

$$\text{f}_\text{F}(0) = 1,\ \text{f}_\text{F}(1) = 0.4,\ \text{f}_\text{F}(2) = 0.5,\ \text{and } \text{f}_\text{F}(3) = 0.2.$$

Then F is a fuzzy UP_s-ideal of A. Since $\text{f}_\text{F}(3 * 2) = \text{f}_\text{F}(1) = 0.4 \ngeq 0.5 = \max\{0.2, 0.5\} = \max\{\text{f}_\text{F}(3), \text{f}_\text{F}(2)\}$, we have F is not a fuzzy UP_i-ideal of A.

Theorem 4.2.24 *The intersection of any nonempty family of fuzzy* UP_s*-ideals of* A *is also a fuzzy* UP_s*-ideal of* A.

Proof. Let F_i be a fuzzy UP_s-ideal of A for all $i \in I$. Then

$$\text{f}_{\bigcap_{i \in I} \text{F}_i}(0) = \inf\{\text{f}_{\text{F}_i}(0)\}_{i \in I}$$

$$\begin{aligned}
&\geq \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}\\
&= \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x),\\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot z)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x\cdot(y\cdot z)),\mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot(y\cdot z))\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot(y\cdot z)),\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\},\ \text{and}\\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x*y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x),\mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x),\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-ideal of A. □

In Example 4.1.21, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follows:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.7	0.5	0.7	0.3
$\mathrm{f}_{\mathrm{F}_2}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-ideals of A. Since $\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3*2) = \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \cup \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-ideal of A.

Theorem 4.2.25 *A nonempty subset S of A is a UP_{s}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{s}-ideal of A.*

Proof. It is straightforward by Theorem 3.0.31 (1) and Theorem 3.0.29 (4). □

Theorem 4.2.26 *The intersection of any nonempty family of fuzzy UP_{i}-ideals of A is also a fuzzy UP_{i}-ideal of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-ideal of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot z)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x\cdot(y\cdot z)), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot(y\cdot z))\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x\cdot(y\cdot z)), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}, \text{ and} \\
\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x * y)\}_{i\in I} \\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigcap_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_i}$-ideal of A. □

In Example 4.2.18, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follows:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.7	0.3	0.4	0.3
$\mathrm{f}_{\mathrm{F}_2}$	0.8	0.5	0.2	0.2

Then F_1 and F_2 are fuzzy $\mathrm{UP_i}$-ideals of A. Since $\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(0\cdot 3) = \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(3) = 0.3 \ngeq 0.4 = \min\{0.4, 0.5\} = \min\{\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(2), \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1)\} = \min\{\mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(0\cdot(1\cdot 3)), \mathrm{f}_{\mathrm{F}_1\cup\mathrm{F}_2}(1)\}$, we have $\mathrm{F}_1 \cup \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_i}$-ideal of A.

Theorem 4.2.27 *A nonempty subset S of A is a UP_{i}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{i}-ideal of A.*

Proof. It is straightforward by Theorem 3.0.31 (2) and Theorem 3.0.29 (4). □

Theorem 4.2.28 *Every fuzzy* UP_s*-ideal of* A *is a fuzzy* UP_s*-filter of* A.

In Example 4.1.16, we define a membership function f_F as follows:

$$f_F(0) = 0.8, \ f_F(1) = 0.6, \ f_F(2) = 0.3, \text{ and } f_F(3) = 0.3.$$

Then F is a fuzzy UP_s-filter of A. Since $f_F(2 \cdot 3) = f_F(2) = 0.3 \ngeq 0.6 = \min\{0.8, 0.6\} = \min\{f_F(0), f_F(1)\} = \min\{f_F(2 \cdot (1 \cdot 3)), f_F(1)\}$, we have F is not a fuzzy UP_s-ideal of A.

Theorem 4.2.29 *Every fuzzy* UP_i*-ideal of* A *is a fuzzy* UP_i*-filter of* A.

In Example 4.1.16, we define a membership function f_F as follows:

$$f_F(0) = 0.8, \ f_F(1) = 0.6, \ f_F(2) = 0.3, \text{ and } f_F(3) = 0.3.$$

Then F is a fuzzy UP_i-filter of A. Since $f_F(2 \cdot 3) = f_F(2) = 0.3 \ngeq 0.6 = \max\{0.8, 0.6\} = \max\{f_F(0), f_F(1)\} = \max\{f_F(2 \cdot (1 \cdot 3)), f_F(1)\}$, we have F is not a fuzzy UP_i-ideal of A.

Definition 4.2.30 A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy strongly* UP_s*-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$, and

(2) a *fuzzy strongly* UP_i*-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Theorem 4.2.31 *Fuzzy strongly* UP_s*-ideals, fuzzy strongly* UP_i*-ideals, and constant fuzzy sets coincide in* A.

Proof. It is straightforward by Theorem 3.0.28. □

If a fuzzy set F_i is constant for all $i \in I$, then we see that the fuzzy sets $\bigcap_{i\in I} \mathrm{F}_i$ and $\bigvee_{i\in I} \mathrm{F}_i$ are constant. From this, we have Theorem 4.2.32 and 4.2.33.

Theorem 4.2.32 *The intersection and union of any nonempty family of fuzzy strongly* $\mathrm{UP_s}$*-ideals of* A *are also a fuzzy strongly* $\mathrm{UP_s}$*-ideal of* A*.*

Theorem 4.2.33 *The intersection and union of any nonempty family of fuzzy strongly* $\mathrm{UP_i}$*-ideals of* A *are also a fuzzy strongly* $\mathrm{UP_i}$*-ideal of* A*.*

Theorem 4.2.34 *A nonempty subset* S *of* A *is a strongly* $\mathrm{UP_s}$*-ideal of* A *if and only if the* t*-characteristic fuzzy set* F_S^t *is a fuzzy strongly* $\mathrm{UP_s}$*-ideal of* A*.*

Proof. It is straightforward by Theorem 3.0.31 (1) and Theorem 3.0.29 (5). □

Theorem 4.2.35 *A nonempty subset* S *of* A *is a strongly* $\mathrm{UP_i}$*-ideal of* A *if and only if the* t*-characteristic fuzzy set* F_S^t *is a fuzzy strongly* $\mathrm{UP_i}$*-ideal of* A*.*

Proof. It is straightforward by Theorem 3.0.31 (2) and Theorem 3.0.29 (5). □

Theorem 4.2.36 *Every fuzzy strongly* $\mathrm{UP_s}$*-ideal (fuzzy strongly* $\mathrm{UP_i}$*-ideal) of* A *is a fuzzy* $\mathrm{UP_s}$*-ideal and a fuzzy* $\mathrm{UP_i}$*-ideal of* A*.*

In Example 4.1.3, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 0.7,\ \mathrm{f_F}(1) = 0.5,\ \mathrm{f_F}(2) = 0.3,\ \text{and } \mathrm{f_F}(3) = 0.6.$$

Then F is a fuzzy $\mathrm{UP_i}$-ideal of A. Since F is not constant, we have F is not a fuzzy strongly $\mathrm{UP_s}$-ideal and a fuzzy strongly $\mathrm{UP_i}$-ideal of A.

Then we get the diagram of generalization of fuzzy sets in fully UP-semigroups as shown in Figure 4.2 below.

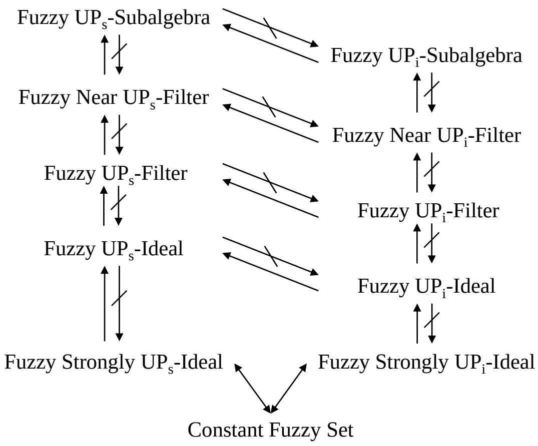

Figure 1: Fuzzy sets in fully UP-semigroups

4.3 Properties of fuzzy sets in UP-algebras

In this section, we shall let A be a UP-algebra $A = (A, \cdot, 0)$ and find some properties of fuzzy sets in UP-algebras.

Proposition 4.3.1 *[34] If* F *is a fuzzy UP-subalgebra of* A*, then*

$$(\forall x \in A)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x)). \tag{4.3.1}$$

Proposition 4.3.2 *If* F *is a fuzzy UP-filter of* A*, then*

$$(\forall x, y \in A)(x \leq y \Rightarrow \mathrm{f_F}(x) \leq \mathrm{f_F}(y)). \tag{4.3.2}$$

Proposition 4.3.3 *If* F *is a fuzzy set in* A *satisfying the condition*

$$(\forall x, y, z \in A)(z \leq x \Rightarrow \mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(y)\}), \tag{4.3.3}$$

then F *is a fuzzy UP-subalgebra of* A*.*

Proof. Let $x, y \in A$. By (3.0.1), we have $x \leq x$. It follows from (4.3.3) that $\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}$. Hence, F is a fuzzy UP-subalgebra of A. □

Theorem 4.3.4 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.3), *then* F *satisfies the condition* (4.3.1).

Proof. It is straightforward by Proposition 4.3.3. □

The following example shows that the converse of Theorem 4.3.4 is not true.

Example 4.3.5 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1, \mathrm{f_F}(1) = 0.6, \mathrm{f_F}(2) = 0.2, \text{ and } \mathrm{f_F}(3) = 0.9.$$

Then F satisfies the condition (4.3.1) but it does not satisfy the condition (4.3.3). Indeed, $1 \leq 1$ but $\mathrm{f_F}(1 \cdot 3) = \mathrm{f_F}(2) = 0.2 \ngeq 0.6 = \min\{0.6, 0.9\} = \min\{\mathrm{f_F}(1), \mathrm{f_F}(3)\}$.

It is clear that we have the following proposition.

Proposition 4.3.6 *If* F *is a fuzzy set in* A *satisfying the condition*

$$(\forall x, y, z \in A)(\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(y)\}), \tag{4.3.4}$$

then F *satisfies the condition* (4.3.3).

The following example shows that the converse of Proposition 4.3.6 is not true.

Example 4.3.7 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1, \mathrm{f_F}(1) = 0.1, \mathrm{f_F}(2) = 0.8, \text{ and } \mathrm{f_F}(3) = 0.2.$$

Then F satisfies the condition (4.3.3) but it does not satisfy the condition (4.3.4). Indeed, $\mathrm{f_F}(1 \cdot 2) = \mathrm{f_F}(3) = 0.2 \not\geq 0.8 = \min\{1, 0.8\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(2)\}$.

Proposition 4.3.8 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.2)*, then* F *is a fuzzy near UP-filter of* A*.*

Proof. Let $x, y \in A$. By (UP-3), we have $x \leq 0$. It follows from (4.3.2) that $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$. By (3.0.5), we have $y \leq x \cdot y$. It follows from (4.3.2) that $\mathrm{f_F}(x \cdot y) \geq \mathrm{f_F}(y)$. Hence, F is a fuzzy near UP-filter of A. □

Theorem 4.3.9 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.2)*, then* F *satisfies the condition* (4.3.4)*.*

Proof. Let $x, y, z \in A$. By (3.0.5), we have $y \leq x \cdot y$. It follows from (4.3.2) that $\mathrm{f_F}(x \cdot y) \geq \mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(y)\}$. Hence, F satisfies (4.3.4). □

The following example shows that the converse of Theorem 4.3.9 is not true.

Example 4.3.10 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1, \mathrm{f_F}(1) = 0.1, \mathrm{f_F}(2) = 0.7, \text{ and } \mathrm{f_F}(3) = 0.8.$$

Then F satisfies the condition (4.3.4) but it does not satisfy the condition (4.3.2). Indeed, $3 \leq 2$ but $\mathrm{f_F}(2) = \mathrm{f_F}(1) = 0.7 \ngeq 0.8 = \mathrm{f_F}(3)$.

Theorem 4.3.11 *If* F *is a fuzzy UP-subalgebra of* A *satisfying the condition*

$$(\forall x, y \in A)(x \cdot y \neq 0 \Rightarrow \mathrm{f_F}(x) \geq \mathrm{f_F}(y)), \tag{4.3.5}$$

then F *is a fuzzy near UP-filter of* A.

Proof. Let $x, y \in A$. If $x \cdot y = 0$, then by (4.3.1), we have $\mathrm{f_F}(x \cdot y) = \mathrm{f_F}(0) \geq \mathrm{f_F}(y)$. If $x \cdot y \neq 0$, then by (4.3.5), we have $\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\} = \mathrm{f_F}(y)$. Hence, F is a fuzzy near UP-filter of A. □

Proposition 4.3.12 *A fuzzy set* F *in* A *satisfies the condition*

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(x)\}) \tag{4.3.6}$$

if and only if F *is a fuzzy UP-filter of* A.

Proof. Let $x \in A$. By (UP-3), we have $x \leq x \cdot 0$. It follows from (4.3.6) that $\mathrm{f_F}(0) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(x)\} = \mathrm{f_F}(x)$. Let $x, y \in A$. By (3.0.1), we have $x \cdot y \leq x \cdot y$. It follows from (4.3.6) that $\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\}$. Hence, F is a fuzzy UP-filter of A.

Conversely, let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $z \cdot (x \cdot y) = 0$, so

$$\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(z \cdot (x \cdot y)), \mathrm{f_F}(z)\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(z)\} = \mathrm{f_F}(z).$$

Thus $\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\} \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(x)\}$. Hence, F satisfies (4.3.6). □

Theorem 4.3.13 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.6)*, then* F *satisfies the condition* (4.3.2).

Proof. Let $x, y \in A$ such that $x \leq y$. By (3.0.11), we have $x \leq x \cdot y$. It follows from (4.3.6) that $\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(x)\} = \mathrm{f_F}(x)$. Hence, F satisfies (4.3.2). □

The following example shows that the converse of Theorem 4.3.13 is not true.

Example 4.3.14 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$f_F(0) = 0.9, f_F(1) = 0.3, f_F(2) = 0.6, \text{ and } f_F(3) = 0.2.$$

Then F satisfies the condition (4.3.2) but it does not satisfy the condition (4.3.6). Indeed, $1 \leq 2 \cdot 3$ but $f_F(3) = 0.2 \not\geq 0.3 = \min\{0.3, 0.6\} = \min\{f_F(1), f_F(2)\}$.

Theorem 4.3.15 *If* F *is a fuzzy near UP-filter of* A *satisfying the condition*

$$(\forall x, y \in A)(f_F(x \cdot y) = f_F(y)), \tag{4.3.7}$$

then F *is a fuzzy UP-filter of* A.

Proof. Let $x, y \in A$. By (4.3.7), we have $f_F(y) \geq \min\{f_F(y), f_F(x)\} = \min\{f_F(x \cdot y), f_F(x)\}$. Hence, F is a fuzzy UP-filter of A. □

Proposition 4.3.16 *A fuzzy set* F *in* A *satisfies the condition*

$$(\forall a, x, y, z \in A)(a \leq x \cdot (y \cdot z) \Rightarrow f_F(x \cdot z) \geq \min\{f_F(a), f_F(y)\}) \tag{4.3.8}$$

if and only if F *is a fuzzy UP-ideal of* A.

Proof. Let $x \in A$. By (UP-3), we have $x \leq x \cdot (x \cdot 0)$. By (UP-3) and (4.3.8), we have

$$f_F(0) = f_F(x \cdot 0) \geq \min\{f_F(x), f_F(x)\} = f_F(x).$$

Let $x, y, z \in A$. By (3.0.1), we have $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (4.3.8) that

$$f_F(x \cdot z) \geq \min\{f_F(x \cdot (y \cdot z)), f_F(y)\}.$$

Hence, F is a fuzzy UP-ideal of A.

Conversely, let $a, x, y, z \in A$ be such that $a \leq x \cdot (y \cdot z)$. By Proposition 4.3.2, we have $\mathrm{f}_{\mathrm{F}}(a) \leq \mathrm{f}_{\mathrm{F}}(x \cdot (y \cdot z))$. Thus

$$\mathrm{f}_{\mathrm{F}}(x \cdot z) \geq \min\{\mathrm{f}_{\mathrm{F}}(x \cdot (y \cdot z)), \mathrm{f}_{\mathrm{F}}(y)\} \geq \min\{\mathrm{f}_{\mathrm{F}}(a), \mathrm{f}_{\mathrm{F}}(y)\}.$$

Hence, F satisfies (4.3.8). □

Proposition 4.3.17 *If* F *is a fuzzy UP-ideal of* A, *then*

$$(\forall a, x, y, z \in A)(a \leq x \cdot (y \cdot z) \Rightarrow \mathrm{f}_{\mathrm{F}}(a \cdot z) \geq \min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\}). \quad (4.3.9)$$

Proof. Let $a, x, y, z \in A$ be such that $a \leq x \cdot (y \cdot z)$. Then $a \cdot (x \cdot (y \cdot z)) = 0$, so

$$\mathrm{f}_{\mathrm{F}}(a \cdot (y \cdot z)) \geq \min\{\mathrm{f}_{\mathrm{F}}(a \cdot (x \cdot (y \cdot z))), \mathrm{f}_{\mathrm{F}}(x)\} = \min\{\mathrm{f}_{\mathrm{F}}(0), \mathrm{f}_{\mathrm{F}}(x)\} = \mathrm{f}_{\mathrm{F}}(x).$$

Thus

$$\mathrm{f}_{\mathrm{F}}(a \cdot z) \geq \min\{\mathrm{f}_{\mathrm{F}}(a \cdot (y \cdot z)), \mathrm{f}_{\mathrm{F}}(y)\} \geq \min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\}.$$

□

Corollary 4.3.18 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.8), *then* F *satisfies the condition* (4.3.9).

Proof. It is straightforward by Propositions 4.3.16 and 4.3.17. □

Theorem 4.3.19 *Let* A *be a UP-algebra satisfying the condition*

$$(\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x)). \quad (4.3.10)$$

If F *is a fuzzy set in* A *satisfying the condition* (4.3.9), *then* F *satisfies the condition* (4.3.8).

Proof. Let $a, x, y, z \in A$ such that $a \leq x \cdot (y \cdot z)$. By (4.3.10), we have $0 =$

$a \cdot (x \cdot (y \cdot z)) = x \cdot (a \cdot (y \cdot z))$, that is, $x \leq a \cdot (y \cdot z)$. It follows from (4.3.9) that $\mathrm{f_F}(x \cdot z) \geq \min\{\mathrm{f_F}(a), \mathrm{f_F}(y)\}$. Hence, F satisfies (4.3.8). □

Theorem 4.3.20 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.9), *then* F *satisfies the condition* (4.3.6).

Proof. Let $x, y, z \in A$ be such that $z \leq x \cdot y$. By (3.0.1) and (3.0.3), we have $0 = z \cdot z \leq z \cdot (x \cdot y)$. By (UP-2) and (4.3.9), we have $\mathrm{f_F}(y) = \mathrm{f_F}(0 \cdot y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(x)\}$. Hence, F satisfies (4.3.6). □

Corollary 4.3.21 *If* F *is a fuzzy set in* A *satisfying the condition* (4.3.8), *then* F *satisfies the condition* (4.3.6).

Proof. It is straightforward by Corollary 4.3.18 and Theorem 4.3.20. □

The following example shows that the converse of Theorem 4.3.20 is not true.

Example 4.3.22 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1, \mathrm{f_F}(1) = 0.9, \mathrm{f_F}(2) = 0.1, \text{ and } \mathrm{f_F}(3) = 0.1.$$

Then F satisfies the condition (4.3.6) but it does not satisfy the condition (4.3.9). Indeed, $3 \leq 1 \cdot (1 \cdot 2)$ but $\mathrm{f_F}(3 \cdot 2) = \mathrm{f_F}(2) = 0.1 \ngeq 0.9 = \mathrm{f_F}(1) = \min\{\mathrm{f_F}(1), \mathrm{f_F}(1)\}$.

The following example shows that fuzzy set in a UP-algebra which satisfies the condition (4.3.8) is not constant.

Example 4.3.23 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. We define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 0.7, \mathrm{f_F}(1) = 0.5, \mathrm{f_F}(2) = 0.4, \text{ and } \mathrm{f_F}(3) = 0.4.$$

Then F satisfies the condition (4.3.8) but it is not constant.

Theorem 4.3.24 *If* F *is a fuzzy UP-filter of* A *satisfying the condition*

$$(\forall x, y, z \in A)(\mathrm{f_F}(y \cdot (x \cdot z)) = \mathrm{f_F}(x \cdot (y \cdot z))), \tag{4.3.11}$$

then F *is a fuzzy UP-ideal of* A.

Proof. Let $x, y, z \in A$. By (4.3.11), we have

$$\mathrm{f_F}(x \cdot z) \geq \min\{\mathrm{f_F}(y \cdot (x \cdot z)), \mathrm{f_F}(y)\} = \min\{\mathrm{f_F}(x \cdot (y \cdot z)), \mathrm{f_F}(y)\}.$$

Hence, F is a fuzzy UP-ideal of A. □

Proposition 4.3.25 *A fuzzy set* F *in* A *satisfies the condition*

$$(\forall a, x, y, z \in A)(a \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{f_F}(x) \geq \min\{\mathrm{f_F}(a), \mathrm{f_F}(y)\}) \tag{4.3.12}$$

if and only if F *is a fuzzy strongly UP-ideal of* A.

Proof. Let $x \in A$. By (UP-3), we have $x \leq 0 = x \cdot 0 = (0 \cdot x) \cdot (0 \cdot 0)$. By (4.3.12), we have $\mathrm{f_F}(0) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(x)\} = \mathrm{f_F}(x)$. Let $x, y, z \in A$. By (3.0.1), we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. By (4.3.12), we have $\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((z \cdot y) \cdot (z \cdot x)), \mathrm{f_F}(y)\}$. Hence, F is a fuzzy strongly UP-ideal of A.

The converse is obvious because F is constant. □

Theorem 4.3.26 *If* F *is a fuzzy set in* A *satisfying the condition*

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{f_F}(z) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}), \tag{4.3.13}$$

then F *satisfies the condition* (4.3.3).

Proof. Let $x, y, z \in A$ be such that $z \leq x$. By (3.0.4), we have $x \cdot y \leq z \cdot y$. By (4.3.13), we have $\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(z), \mathrm{f_F}(y)\}$. Hence, F satisfies (4.3.3). □

Proposition 4.3.27 *A fuzzy set* F *in* A *satisfies the condition* (4.3.13) *if and only if* F *is a fuzzy strongly UP-ideal of* A.

Proof. Let $x \in A$. By (UP-3), we have $x \leq 0 = 0 \cdot 0$. By (4.3.13), we have $\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}(0), \mathrm{f_F}(0)\} = \mathrm{f_F}(0)$. By Theorem 4.3.26 and Proposition 4.3.3, we have $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$. Thus $\mathrm{f_F}(x) = \mathrm{f_F}(0)$ for all $x \in A$, so F is constant. Hence, F a fuzzy strongly UP-ideal of A.

The converse is obvious because F is constant. □

Theorem 4.3.28 *If* F *is a fuzzy set in* A *satisfying the condition*

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{f_F}(z) \geq \mathrm{f_F}(y)), \tag{4.3.14}$$

then F *satisfies the condition* (4.3.3).

Proof. Let $x, y, z \in A$ be such that $z \leq x$. By (3.0.4), we have $x \cdot y \leq z \cdot y$. It follows from (4.3.14) that $\mathrm{f}_{\mathrm{F}}(x \cdot y) \geq \mathrm{f}_{\mathrm{F}}(y) \geq \min\{\mathrm{f}_{\mathrm{F}}(z), \mathrm{f}_{\mathrm{F}}(y)\}$. Hence, F satisfies (4.3.3). □

Proposition 4.3.29 *A fuzzy set* F *in* A *satisfies the condition* (4.3.14) *if and only if* F *is a fuzzy strongly UP-ideal of* A*.*

Proof. Let $x \in A$. By (UP-3), we have $x \leq 0 = 0 \cdot 0$. By (4.3.14), we have $\mathrm{f}_{\mathrm{F}}(x) \geq \mathrm{f}_{\mathrm{F}}(0)$. By Theorem 4.3.28 and Proposition 4.3.3, we have $\mathrm{f}_{\mathrm{F}}(0) \geq \mathrm{f}_{\mathrm{F}}(x)$. Thus $\mathrm{f}_{\mathrm{F}}(x) = \mathrm{f}_{\mathrm{F}}(0)$ for all $x \in A$, so F is constant. Hence, F is a fuzzy strongly UP-ideal of A.

The converse is obvious because F is constant. □

We have provided various important properties of fuzzy sets in various types in UP-algebras which will be used in the next section. We get the diagram of the properties of fuzzy sets in UP-algebras as shown in Figure 4.3 below.

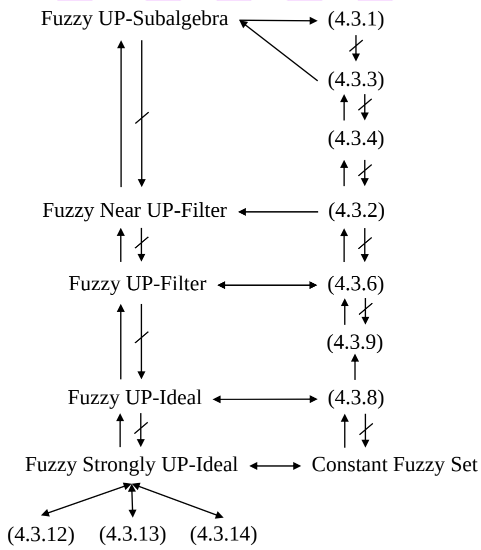

Figure 2: Properties of fuzzy sets in UP-algebras

4.4 Fuzzy soft sets over fully UP-semigroups

From now on, we shall let A be an f-UP-semigroup $A = (A, \cdot, *, 0)$ and P be a set of parameters. Let $\mathcal{F}(A)$ denotes the set of all fuzzy sets in A. A subset E of P is called a *set of statistics*.

Definition 4.4.1 Let $E \subseteq P$. A pair $(\widetilde{\mathrm{F}}, E)$ is called a *fuzzy soft set* over A if $\widetilde{\mathrm{F}}$ is a mapping given by $\widetilde{\mathrm{F}} \colon E \to \mathcal{F}(A)$, that is, a fuzzy soft set is a statistic family of fuzzy sets in A. In general, for every $e \in E$, $\widetilde{\mathrm{F}}[e] := \{(x, \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x)) \mid x \in A\}$ is a fuzzy set in A and it is called a *fuzzy value set* of statistic e.

Definition 4.4.2 Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft sets over a common universe U. The *union* [24] of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \cup (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \cup E_2$ and

(ii) for all $e \in E$,

$$\widetilde{\mathrm{H}}[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E_1 \setminus E_2 \\ \widetilde{\mathrm{G}}[e] & \text{if } e \in E_2 \setminus E_1 \\ \widetilde{\mathrm{F}}[e] \cup \widetilde{\mathrm{G}}[e] & \text{if } e \in E_1 \cap E_2. \end{cases}$$

The *restricted union* [28] of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \Cup (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \cap E_2 \neq \emptyset$ and

(ii) $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cup \widetilde{\mathrm{G}}[e]$ for all $e \in E$.

Definition 4.4.3 Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft sets over a common universe U. The *extended intersection* [28] of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \cap (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \cup E_2$ and

(ii) for all $e \in E$,

$$\widetilde{\mathrm{H}}[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E_1 \setminus E_2 \\ \widetilde{\mathrm{G}}[e] & \text{if } e \in E_2 \setminus E_1 \\ \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e] & \text{if } e \in E_1 \cap E_2. \end{cases}$$

The *intersection* [2] of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \Cap (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \cap E_2 \neq \emptyset$ and

(ii) $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e]$ for all $e \in E$.

Definition 4.4.4 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft* UP_s*-subalgebra* based on $e \in E$ (we shortly call an *e-fuzzy soft* UP_s*-subalgebra*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy $\mathrm{UP_s}$-subalgebra of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft* UP_s*-subalgebra* of A.

In the next theorem, we give necessary condition for fuzzy soft $\mathrm{UP_s}$-subalgebras of f-UP-semigroups.

Theorem 4.4.5 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft set over* A *such that for all* $e \in E$*, a fuzzy set* $\widetilde{\mathrm{F}}[e]$ *in* A *satisfies the conditions* (4.3.3) *and* (3.0.14)*, then* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_s*-subalgebra of* A*.*

Proof. It is straightforward by Proposition 4.3.3 and Lemma 3.0.36 (1). □

The proof of the following theorem can be verified easily.

Theorem 4.4.6 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_s*-subalgebra of* A *and* $\emptyset \neq E^* \subseteq E$*, then* $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ *is a fuzzy soft* UP_s*-subalgebra of* A*.*

The following example shows that there exists a nonempty subset E^* of E such that $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A, but $(\widetilde{\mathrm{F}}, E)$ is not a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

Example 4.4.7 Let A be the set of four series of the iPhone, that is,

$$A = \{5, 6, 7, \mathrm{X}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	X	7	6	5
X	X	7	6	5
7	X	X	6	5
6	X	7	X	5
5	X	7	6	X

$*$	X	7	6	5
X	X	X	X	X
7	X	X	X	X
6	X	X	X	7
5	X	X	7	X

Then $A = (A, \cdot, *, \mathrm{X})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over A where

$$E := \{\text{price, beauty, specifications, stability}\}$$

with $\widetilde{\mathrm{F}}[\text{price}], \widetilde{\mathrm{F}}[\text{beauty}], \widetilde{\mathrm{F}}[\text{specifications}]$, and $\widetilde{\mathrm{F}}[\text{stability}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}$	X	7	6	5
price	0.8	0.3	0.7	0.1
beauty	0.5	0.3	0.2	0.4
specifications	0.9	0.8	0.5	0.6
stability	1	0.4	0.7	0.6

Then $\widetilde{\mathrm{F}}[\text{stability}]$ is not a fuzzy $\mathrm{UP_s}$-subalgebra of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\text{stability}]}(5 * 6) = \mathrm{f}_{\widetilde{\mathrm{F}}[\text{stability}]}(7) = 0.4 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{stability}]}(5), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{stability}]}(6)\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a fuzzy soft $\mathrm{UP_s}$-subalgebra of A. We take

$$E^* := \{\text{price, beauty, specifications}\}.$$

Thus $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

Theorem 4.4.8 *The extended intersection of two fuzzy soft UP_s-subalgebras of A is also a fuzzy soft UP_s-subalgebra. Moreover, the intersection of two fuzzy soft UP_s-subalgebras of A is also a fuzzy soft UP_s-subalgebra.*

Proof. Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft $\mathrm{UP_s}$-subalgebras of A. Assume that $(\widetilde{\mathrm{F}}, E_1) \cap (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ with $E = E_1 \cup E_2$. Let $e \in E$.

Case 1: $e \in E_1 \setminus E_2$ (resp., $e \in E_2 \setminus E_1$). Then $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e]$ (resp., $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{G}}[e]$) is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

Case 2: $e \in E_1 \cap E_2$. By Theorem 4.2.2, we have $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e]$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra.

Thus $(\widetilde{\mathrm{H}}, E)$ is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra of A for all $e \in E$. Hence, $(\widetilde{\mathrm{H}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A. □

Theorem 4.4.9 *The union of two fuzzy soft UP_s-subalgebras of A is also a fuzzy soft UP_s-subalgebra if sets of statistics of two fuzzy soft UP_s-subalgebras are disjoint.*

Proof. Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft $\mathrm{UP_s}$-subalgebras of A such that $E_1 \cap E_2 = \emptyset$. Assume that $(\widetilde{\mathrm{F}}, E_1) \cup (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ with $E = E_1 \cup E_2$. Let $e \in E$. Since $E_1 \cap E_2 = \emptyset$, we have $e \in E_1 \setminus E_2$ or $e \in E_2 \setminus E_1$.

Case 1: $e \in E_1 \setminus E_2$. Then $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e]$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

Case 2: $e \in E_2 \setminus E_1$. Then $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{G}}[e]$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

Thus $(\widetilde{\mathrm{H}}, E)$ is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra of A for all $e \in E$. Hence, $(\widetilde{\mathrm{H}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A. □

The following example shows that Theorem 4.4.9 is not valid if sets of statistics of two fuzzy soft $\mathrm{UP_s}$-subalgebras are not disjoint.

Example 4.4.10 By Cayley tables in Example 4.4.7, we know that $A = (A, \cdot, *, \mathrm{X})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ be two fuzzy soft sets over A where

$$E_1 := \{\text{price, beauty, specifications}\} \text{ and } E_2 := \{\text{price, stability}\}$$

with $\widetilde{\mathrm{G}}_1[\text{price}], \widetilde{\mathrm{G}}_1[\text{beauty}], \widetilde{\mathrm{G}}_1[\text{specifications}], \widetilde{\mathrm{G}}_2[\text{price}]$, and $\widetilde{\mathrm{G}}_2[\text{stability}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{G}}_1$	X	7	6	5
price	0.9	0.7	0.9	0.2
beauty	1	0.8	0.3	0.2
specifications	0.6	0.5	0.3	0.4

$\widetilde{\mathrm{G}}_2$	X	7	6	5
price	0.9	0.3	0.2	0.8
stability	0.7	0.2	0.5	0.2

Then $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_s}$-subalgebras of A. Since price $\in E_1 \cap E_2$, we have

$$(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{price}] \cup \widetilde{\mathrm{G}}_2[\text{price}]})(6 * 5) = (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{price}] \cup \widetilde{\mathrm{G}}_2[\text{price}]})(7)$$

$$= 0.7$$
$$\ngeq 0.8$$
$$= \min\{0.9, 0.8\}$$
$$= \min\{(f_{\widetilde{G}_1[\text{price}] \cup \widetilde{G}_2[\text{price}]})(6), (f_{\widetilde{G}_1[\text{price}] \cup \widetilde{G}_2[\text{price}]})(5)\}.$$

Thus $\widetilde{G}_1[\text{price}] \cup \widetilde{G}_2[\text{price}]$ is not a fuzzy UP_s-subalgebra of A, that is, $(\widetilde{G}_1, E_1) \cup (\widetilde{G}_2, E_2)$ is not a price-fuzzy soft UP_s-subalgebra of A. Hence, $(\widetilde{G}_1, E_1) \cup (\widetilde{G}_2, E_2)$ is not a fuzzy soft UP_s-subalgebra of A. Moreover, $(\widetilde{G}_1, E_1) \Cup (\widetilde{G}_2, E_2)$ is not a fuzzy soft UP_s-subalgebra of A.

Definition 4.4.11 A fuzzy soft set $(\widetilde{F}, E)$ over A is called a *fuzzy soft UP_i-subalgebra* based on $e \in E$ (we shortly call an *e-fuzzy soft UP_i-subalgebra*) of A if a fuzzy set $\widetilde{F}[e]$ in A is a fuzzy UP_i-subalgebra of A. If $(\widetilde{F}, E)$ is an e-fuzzy soft UP_i-subalgebra of A for all $e \in E$, we say that $(\widetilde{F}, E)$ is a *fuzzy soft UP_i-subalgebra* of A.

In the next theorem, we give necessary condition for fuzzy soft UP_i-subalgebras of f-UP-semigroups.

Theorem 4.4.12 *If $(\widetilde{F}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{F}[e]$ in A satisfies the conditions* (4.3.3) *and* (3.0.15)*, then $(\widetilde{F}, E)$ is a fuzzy soft UP_i-subalgebra of A.*

Proof. It is straightforward by Proposition 4.3.3 and Lemma 3.0.36 (2). □

Theorem 4.4.13 *Every e-fuzzy soft UP_i-subalgebra of A is an e-fuzzy soft UP_s-subalgebra. Moreover, every fuzzy soft UP_i-subalgebra of A is a fuzzy soft UP_s-subalgebra.*

The following example shows that the converse of Theorem 4.4.13 is not true.

Example 4.4.14 In Example 4.4.7, we know that $(\widetilde{\mathrm{F}}, E)$ is a price-fuzzy soft $\mathrm{UP_s}$-subalgebra of A but $\widetilde{\mathrm{F}}[\text{price}]$ is not a fuzzy $\mathrm{UP_i}$-subalgebra of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\text{price}]}(6 * 5) = \mathrm{f}_{\widetilde{\mathrm{F}}[\text{price}]}(7) = 0.3 \ngeq 0.7 = \max\{0.7, 0.1\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{price}]}(6), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{price}]}(5)\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a price-fuzzy soft $\mathrm{UP_i}$-subalgebra of A.

The proof of the following theorem can be verified easily.

Theorem 4.4.15 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP_{i}-subalgebra of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft UP_{i}-subalgebra of A.*

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.16 *The extended intersection of two fuzzy soft UP_{i}-subalgebras of A is also a fuzzy soft UP_{i}-subalgebra. Moreover, the intersection of two fuzzy soft UP_{i}-subalgebras of A is also a fuzzy soft UP_{i}-subalgebra.*

Theorem 4.4.17 *The union of two fuzzy soft UP_{i}-subalgebras of A is also a fuzzy soft UP_{i}-subalgebra if sets of statistics of two fuzzy soft UP_{i}-subalgebras are disjoint.*

The following example shows that Theorem 4.4.17 is not valid if sets of statistics of two fuzzy soft $\mathrm{UP_i}$-subalgebras are not disjoint.

Example 4.4.18 Let A be the set of four types of a music, that is,

$$A = \{\text{pop, rock, classic, disco}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	pop	rock	disco	classic
pop	pop	rock	disco	classic
rock	pop	pop	disco	disco
disco	pop	rock	pop	disco
classic	pop	rock	pop	pop

$*$	pop	rock	disco	classic
pop	pop	pop	pop	pop
rock	pop	pop	pop	pop
disco	pop	pop	pop	pop
classic	pop	pop	pop	pop

Then $A = (A, \cdot, *, \text{pop})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ be two fuzzy soft sets over A where

$$E_1 := \{\text{sorrow, modernity}\} \text{ and } E_2 := \{\text{modernity, enjoyment}\}$$

with $\widetilde{\mathrm{G}}_1[\text{sorrow}], \widetilde{\mathrm{G}}_1[\text{modernity}], \widetilde{\mathrm{G}}_2[\text{modernity}]$, and $\widetilde{\mathrm{G}}_2[\text{enjoyment}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{G}}_1$	pop	rock	disco	classic
sorrow	0.7	0.7	0.5	0.5
modernity	0.9	0.8	0.3	0.3

$\widetilde{\mathrm{G}}_2$	pop	rock	disco	classic
modernity	0.8	0.3	0.4	0.5
enjoyment	1	0.9	0.1	0.1

Then $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_i}$-subalgebras of A. Since modernity $\in E_1 \cap E_2$, we have

$$
\begin{aligned}
&(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{modernity}] \cup \widetilde{\mathrm{G}}_2[\text{modernity}]})(\text{rock} \cdot \text{classic}) \\
&\quad = (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{modernity}] \cup \widetilde{\mathrm{G}}_2[\text{modernity}]})(\text{disco}) \\
&\quad = 0.4 \\
&\quad \ngeq 0.5 \\
&\quad = \min\{0.8, 0.5\} \\
&\quad = \min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{modernity}] \cup \widetilde{\mathrm{G}}_2[\text{modernity}]})(\text{rock}), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{modernity}] \cup \widetilde{\mathrm{G}}_2[\text{modernity}]})(\text{classic})\}.
\end{aligned}
$$

Thus $\widetilde{\mathrm{G}}_1[\text{modernity}] \cup \widetilde{\mathrm{G}}_2[\text{modernity}]$ is not a fuzzy $\mathrm{UP_i}$-subalgebra of A, that is, $(\widetilde{\mathrm{G}}_1, E_1) \cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a modernity-fuzzy soft $\mathrm{UP_i}$-subalgebra of A. Hence, $(\widetilde{\mathrm{G}}_1, E_1) \cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_i}$-subalgebra of A. Moreover, $(\widetilde{\mathrm{G}}_1, E_1) \Cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_i}$-subalgebra of A.

Definition 4.4.19 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft near* $\mathit{UP_s}$*-filter* based on $e \in E$ (we shortly call an *e-fuzzy soft near* $\mathit{UP_s}$*-filter*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy near $\mathrm{UP_s}$-filter of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft near $\mathrm{UP_s}$-filter of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft near* $\mathit{UP_s}$*-filter* of A.

In the next theorem, we give necessary condition for fuzzy soft near $\mathrm{UP_s}$-filters of f-UP-semigroups.

Theorem 4.4.20 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft set over* A *such that for all* $e \in E$*, a fuzzy set* $\widetilde{\mathrm{F}}[e]$ *in* A *satisfies the conditions* (4.3.2) *and* (3.0.14)*, then* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft near* $\mathit{UP_s}$*-filter of* A.

Proof. It is straightforward by Proposition 4.3.8 and Lemma 3.0.36 (1). □

Theorem 4.4.21 *Every e-fuzzy soft near* $\mathit{UP_s}$*-filter of* A *is an e-fuzzy soft* $\mathit{UP_s}$-

subalgebra. Moreover, every fuzzy soft near UP_s*-filter of* A *is a fuzzy soft* UP_s*-subalgebra.*

The following example shows that the converse of Theorem 4.4.21 is not true.

Example 4.4.22 Let A be a set of four foods, that is,

$$A = \{\text{apple, banana, meat, rice}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	rice	apple	banana	meat
rice	rice	apple	banana	meat
apple	rice	rice	apple	meat
banana	rice	rice	rice	meat
meat	rice	apple	apple	rice

$*$	rice	apple	banana	meat
rice	rice	rice	rice	rice
apple	rice	rice	rice	rice
banana	rice	rice	rice	rice
meat	rice	rice	rice	apple

Then $A = (A, \cdot, *, \text{rice})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over A where

$$E := \{\text{pig, monkey, chicken}\}$$

with $\widetilde{\mathrm{F}}[\mathrm{pig}], \widetilde{\mathrm{F}}[\mathrm{monkey}]$, and $\widetilde{\mathrm{F}}[\mathrm{chicken}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}$	rice	apple	banana	meat
pig	1	0.8	0.9	0.3
monkey	0.8	0.4	0.8	0.3
chicken	0.7	0.4	0.3	0.2

Then $(\widetilde{\mathrm{F}}, E)$ is a pig-fuzzy soft $\mathrm{UP_s}$-subalgebra of A. But $(\widetilde{\mathrm{F}}, E)$ is not a pig-fuzzy soft near $\mathrm{UP_s}$-filter of A since

$$\begin{aligned} f_{\widetilde{\mathrm{F}}[\mathrm{pig}]}(\mathrm{meat} \cdot \mathrm{banana}) &= f_{\widetilde{\mathrm{F}}[\mathrm{pig}]}(\mathrm{apple}) \\ &= 0.8 \\ &\ngeq 0.9 \\ &= f_{\widetilde{\mathrm{F}}[\mathrm{pig}]}(\mathrm{banana}), \end{aligned}$$

that is, $\widetilde{\mathrm{F}}[\mathrm{pig}]$ is not a fuzzy near $\mathrm{UP_s}$-filter of A.

In the next theorem, we give necessary condition for fuzzy soft $\mathrm{UP_s}$-subalgebras as fuzzy soft near $\mathrm{UP_s}$-filters of f-UP-semigroups.

Theorem 4.4.23 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition* (4.3.5)*, then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A.*

Proof. It is straightforward by Theorem 4.3.11. □

The proof of the following theorem can be verified easily.

Theorem 4.4.24 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A.*

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.25 *The extended intersection of two fuzzy soft near* UP_s*-filters of* A *is also a fuzzy soft near* UP_s*-filter. Moreover, the intersection of two fuzzy soft near* UP_s*-filters of* A *is also a fuzzy soft near* UP_s*-filter.*

Theorem 4.4.26 *The union of two fuzzy soft near* UP_s*-filters of* A *is also a fuzzy soft near* UP_s*-filter if sets of statistics of two fuzzy soft near* UP_s*-filters are disjoint.*

The following example shows that Theorem 4.4.26 is not valid if sets of statistics of two fuzzy soft near UP_s-filters are not disjoint.

Example 4.4.27 In Example 4.4.10, we have $(\widetilde{G}_1, E_1)$ and $(\widetilde{G}_2, E_2)$ are two fuzzy soft near UP_s-filters of A. Since price $\in E_1 \cap E_2$, we have

$$\begin{aligned}(f_{\widetilde{G}_1[\text{price}]\cup\widetilde{G}_2[\text{price}]})(6 * 5) &= (f_{\widetilde{G}_1[\text{price}]\cup\widetilde{G}_2[\text{price}]})(7)\\ &= 0.7\\ &\not\geq 0.8\\ &= \min\{0.9, 0.8\}\\ &= \min\{(f_{\widetilde{G}_1[\text{price}]\cup\widetilde{G}_2[\text{price}]})(6), (f_{\widetilde{G}_1[\text{price}]\cup\widetilde{G}_2[\text{price}]})(5)\}.\end{aligned}$$

Thus $\widetilde{G}_1[\text{price}] \cup \widetilde{G}_2[\text{price}]$ is not a fuzzy near UP_s-filter of A, that is, $(\widetilde{G}_1, E_1) \cup (\widetilde{G}_2, E_2)$ is not a price-fuzzy soft near UP_s-filter of A. Hence, $(\widetilde{G}_1, E_1) \cup (\widetilde{G}_2, E_2)$ is not a fuzzy soft near UP_s-filter of A. Moreover, $(\widetilde{G}_1, E_1) \Cup (\widetilde{G}_2, E_2)$ is not a fuzzy soft near UP_s-filter of A.

Definition 4.4.28 A fuzzy soft set $(\widetilde{F}, E)$ over A is called a *fuzzy soft near* UP_i*-filter* based on $e \in E$ (we shortly call an e*-fuzzy soft near* UP_i*-filter*) of A if a fuzzy set $\widetilde{F}[e]$ in A is a fuzzy near UP_i-filter of A. If $(\widetilde{F}, E)$ is an e-fuzzy soft near

$\mathrm{UP_i}$-filter of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft near UP_{i}-filter* of A.

In the next theorem, we give necessary condition for fuzzy soft near $\mathrm{UP_i}$-filters of f-UP-semigroups.

Theorem 4.4.29 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.2) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near UP_{i}-filter of A.*

Proof. It is straightforward by Proposition 4.3.8 and Lemma 3.0.36 (2). □

Theorem 4.4.30 *Every e-fuzzy soft near UP_{i}-filter of A is an e-fuzzy soft near UP_{s}-filter. Moreover, every fuzzy soft near UP_{i}-filter of A is a fuzzy soft near UP_{s}-filter.*

Theorem 4.4.31 *Every e-fuzzy soft near UP_{i}-filter of A is an e-fuzzy soft UP_{i}-subalgebra. Moreover, every fuzzy soft near UP_{i}-filter of A is a fuzzy soft UP_{i}-subalgebra.*

The following two examples show that the converse of Theorems 4.4.30 and 4.4.31 is not true.

Example 4.4.32 In Example 4.4.7, we know that $(\widetilde{\mathrm{F}}, E)$ is a price-fuzzy soft near $\mathrm{UP_s}$-filter of A but $\widetilde{\mathrm{F}}[\mathrm{price}]$ is not a fuzzy near $\mathrm{UP_i}$-filter of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(6 * 5) = \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(7) = 0.3 \not\geq 0.7 = \max\{0.7, 0.1\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(6), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(5)\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a price-fuzzy soft near $\mathrm{UP_i}$-filter of A.

Example 4.4.33 In Example 4.4.22, we know that $(\widetilde{\mathrm{F}}, E)$ is a monkey-fuzzy soft $\mathrm{UP_i}$-subalgebra of A but $\widetilde{\mathrm{F}}[\mathrm{monkey}]$ is not a fuzzy near $\mathrm{UP_i}$-filter of A. Indeed,

$$f_{\widetilde{F}[\text{monkey}]}(\text{apple} \cdot \text{banana}) = f_{\widetilde{F}[\text{monkey}]}(\text{apple}) = 0.4 \not\geq 0.8 = f_{\widetilde{F}[\text{monkey}]}(\text{banana}).$$

Hence, $(\widetilde{F}, E)$ is not a monkey-fuzzy soft near UP_i-filter of A.

In the next theorem, we give necessary condition for fuzzy soft UP_i-subalgebras as fuzzy soft near UP_i-filters of f-UP-semigroups.

Theorem 4.4.34 *If $(\widetilde{F}, E)$ is a fuzzy soft UP_i-subalgebra of A such that for all $e \in E$, a fuzzy set $\widetilde{F}[e]$ in A satisfies the condition (4.3.5), then $(\widetilde{F}, E)$ is a fuzzy soft near UP_i-filter of A.*

Proof. It is straightforward by Theorem 4.3.11. □

The proof of the following theorem can be verified easily.

Theorem 4.4.35 *If $(\widetilde{F}, E)$ is a fuzzy soft near UP_i-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{F}|_{E^*}, E^*)$ is a fuzzy soft near UP_i-filter of A.*

By using Theorem 4.2.10, we can obtain the following two theorems in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.36 *The extended intersection of two fuzzy soft near UP_i-filters of A is also a fuzzy soft near UP_i-filter. Moreover, the intersection of two fuzzy soft near UP_i-filters of A is also a fuzzy soft near UP_i-filter.*

Theorem 4.4.37 *The union of two fuzzy soft near UP_i-filters of A is also a fuzzy soft near UP_i-filter. Moreover, the restricted union of two fuzzy soft near UP_i-filters of A is also a fuzzy soft near UP_i-filter.*

Definition 4.4.38 A fuzzy soft set $(\widetilde{F}, E)$ over A is called a *fuzzy soft UP_s-filter* based on $e \in E$ (we shortly call an *e-fuzzy soft UP_s-filter*) of A if a fuzzy set $\widetilde{F}[e]$ in A is a fuzzy UP_s-filter of A. If $(\widetilde{F}, E)$ is an e-fuzzy soft UP_s-filter of A for all $e \in E$, we say that $(\widetilde{F}, E)$ is a *fuzzy soft UP_s-filter* of A.

In the next theorem, we give necessary condition for fuzzy soft $\mathrm{UP_s}$-filters of f-UP-semigroups.

Theorem 4.4.39 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft set over* A *such that for all* $e \in E$*, a fuzzy set* $\widetilde{\mathrm{F}}[e]$ *in* A *satisfies the conditions* (4.3.6) *and* (3.0.14)*, then* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_{s}*-filter of* A*.*

Proof. It is straightforward by Proposition 4.3.12 and Lemma 3.0.36 (1). □

Theorem 4.4.40 *Every e-fuzzy soft* UP_{s}*-filter of* A *is an e-fuzzy soft near* UP_{s}*-filter. Moreover, every fuzzy soft* UP_{s}*-filter of* A *is a fuzzy soft near* UP_{s}*-filter.*

The following example shows that the converse of Theorem 4.4.40 is not true.

Example 4.4.41 Let A be a set of four coffees, that is,

$$A = \{\text{Mocha(M), Americano(A), Cappuccino(C), Latte(L)}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	L	A	M	C
L	L	A	M	C
A	L	L	M	C
M	L	L	L	C
C	L	L	L	L

$*$	L	A	M	C
L	L	L	L	L
A	L	L	L	L
M	L	L	L	L
C	L	L	L	M

Then $A = (A, \cdot, *, \text{Latte})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over A where

$$E := \{\text{sweetness, strong, aroma}\}$$

with $\widetilde{\mathrm{F}}[\text{sweetness}], \widetilde{\mathrm{F}}[\text{strong}]$, and $\widetilde{\mathrm{F}}[\text{aroma}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}$	L	A	M	C
sweetness	0.8	0.1	0.6	0.6
strong	0.7	0.7	0.6	0.5
aroma	0.5	0.3	0.4	0.1

Then $(\widetilde{\mathrm{F}}, E)$ is a sweetness-fuzzy soft near UP$_\mathrm{s}$-filter of A but $\widetilde{\mathrm{F}}[\text{sweetness}]$ is not a fuzzy UP$_\mathrm{s}$-filter of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\text{sweetness}]}(\mathrm{A}) = 0.1 \ngeq 0.6 = \min\{0.8, 0.6\} =$$
$$\min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{sweetness}]}(\mathrm{L}), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{sweetness}]}(\mathrm{M})\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{sweetness}]}(\mathrm{M} \cdot \mathrm{A}), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{sweetness}]}(\mathrm{M})\}$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a sweetness-fuzzy soft UP$_\mathrm{s}$-filter of A.

In the next theorem, we give necessary condition for fuzzy soft near UP$_\mathrm{s}$-filters as fuzzy soft UP$_\mathrm{s}$-filters of f-UP-semigroups.

Theorem 4.4.42 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near UP$_\mathrm{s}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition* (4.3.7), *then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{s}$-filter of A.*

Proof. It is straightforward by Theorem 4.3.15. □

The proof of the following theorem can be verified easily.

Theorem 4.4.43 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{s}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft UP$_\mathrm{s}$-filter of A.*

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.44 *The extended intersection of two fuzzy soft UP$_{\text{s}}$-filters of A is also a fuzzy soft UP$_{\text{s}}$-filter. Moreover, the intersection of two fuzzy soft UP$_{\text{s}}$-filters of A is also a fuzzy soft UP$_{\text{s}}$-filter.*

Theorem 4.4.45 *The union of two fuzzy soft UP$_{\text{s}}$-filters of A is also a fuzzy soft UP$_{\text{s}}$-filter if sets of statistics of two fuzzy soft UP$_{\text{s}}$-filters are disjoint.*

The following example shows that Theorem 4.4.45 is not valid if sets of statistics of two fuzzy soft UP$_{\text{s}}$-filters are not disjoint.

Example 4.4.46 In Example 4.4.10, we have $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft UP$_{\text{s}}$-filters of A. Since price $\in E_1 \cap E_2$, we have

$$(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(6 * 5) = (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(7) = 0.7 \ngeq 0.8 = \min\{0.9, 0.8\} = \min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(6), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(5)\}.$$

Thus $\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]$ is not a fuzzy UP$_{\text{s}}$-filter of A, that is, $(\widetilde{\mathrm{G}}_1, E_1)\cup(\widetilde{\mathrm{G}}_2, E_2)$ is not a price-fuzzy soft UP$_{\text{s}}$-filter of A. Hence, $(\widetilde{\mathrm{G}}_1, E_1) \cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft UP$_{\text{s}}$-filter of A. Moreover, $(\widetilde{\mathrm{G}}_1, E_1) \Cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft UP$_{\text{s}}$-filter of A.

Definition 4.4.47 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft UP$_{\text{i}}$-filter* based on $e \in E$ (we shortly call an *e-fuzzy soft UP$_{\text{i}}$-filter*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy UP$_{\text{i}}$-filter of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft UP$_{\text{i}}$-filter of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft UP$_{\text{i}}$-filter* of A.

In the next theorem, we give necessary condition for fuzzy soft UP$_{\text{i}}$-filters of f-UP-semigroups.

Theorem 4.4.48 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.6) *and* (3.0.15)*, then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_{\text{i}}$-filter of A.*

Proof. It is straightforward by Proposition 4.3.12 and Lemma 3.0.36 (2). □

Theorem 4.4.49 *Every e-fuzzy soft UP$_\mathrm{i}$-filter of A is an e-fuzzy soft UP$_\mathrm{s}$-filter. Moreover, every fuzzy soft UP$_\mathrm{i}$-filter of A is a fuzzy soft UP$_\mathrm{s}$-filter.*

Theorem 4.4.50 *Every e-fuzzy soft UP$_\mathrm{i}$-filter of A is an e-fuzzy soft near UP$_\mathrm{i}$-filter. Moreover, every fuzzy soft UP$_\mathrm{i}$-filter of A is a fuzzy soft near UP$_\mathrm{i}$-filter.*

The following two examples show that the converse of Theorems 4.4.49 and 4.4.50 is not true.

Example 4.4.51 In Example 4.4.7, we know that $(\widetilde{\mathrm{F}}, E)$ is a beauty-fuzzy soft UP$_\mathrm{s}$-filter of A but $\widetilde{\mathrm{F}}$[beauty] is not a fuzzy UP$_\mathrm{i}$-filter of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{beauty}]}(6 * 5) = \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{beauty}]}(7) = 0.3 \ngeq 0.4 = \max\{0.2, 0.4\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{beauty}]}(6), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{beauty}]}(5)\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a beauty-fuzzy soft UP$_\mathrm{i}$-filter of A.

Example 4.4.52 In Example 4.4.41, we know that $(\widetilde{\mathrm{F}}, E)$ is a aroma-fuzzy soft near UP$_\mathrm{i}$-filter of A but $\widetilde{\mathrm{F}}$[aroma] is not a fuzzy UP$_\mathrm{i}$-filter of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{aroma}]}(\mathrm{A}) = 0.3 \ngeq 0.4 = \min\{0.5, 0.4\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{aroma}]}(\mathrm{L}), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{aroma}]}(\mathrm{M})\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{aroma}]}(\mathrm{M} \cdot \mathrm{A}), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{aroma}]}(\mathrm{M})\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a aroma-fuzzy soft UP$_\mathrm{i}$-filter of A.

In the next theorem, we give necessary condition for fuzzy soft near UP$_\mathrm{i}$-filters as fuzzy soft UP$_\mathrm{i}$-filters of f-UP-semigroups.

Theorem 4.4.53 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near UP$_\mathrm{i}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.7), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{i}$-filter of A.*

Proof. It is straightforward by Theorem 4.3.15. □

The proof of the following theorem can be verified easily.

Theorem 4.4.54 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_{i}*-filter of* A *and* $\emptyset \neq E^* \subseteq E$*, then* $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ *is a fuzzy soft* UP_{i}*-filter of* A.

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.55 *The extended intersection of two fuzzy soft* UP_{i}*-filters of* A *is also a fuzzy soft* UP_{i}*-filter. Moreover, the intersection of two fuzzy soft* UP_{i}*-filters of* A *is also a fuzzy soft* UP_{i}*-filter.*

Theorem 4.4.56 *The union of two fuzzy soft* UP_{i}*-filters of* A *is also a fuzzy soft* UP_{i}*-filter if sets of statistics of two fuzzy soft* UP_{i}*-filters are disjoint.*

The following example shows that Theorem 4.4.56 is not valid if sets of statistics of two fuzzy soft UP_{i}-filters are not disjoint.

Example 4.4.57 Let A be a set of four colors, that is,

$$A = \{\text{blue, green, cyan, black}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	black	cyan	blue	green
black	black	cyan	blue	green
cyan	black	black	blue	blue
blue	black	cyan	black	cyan
green	black	black	black	black

$*$	black	cyan	blue	green
black	black	black	black	black
cyan	black	black	black	black
blue	black	black	black	black
green	black	black	black	black

Then $A = (A, \cdot, *, \text{black})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ be two fuzzy soft sets over A where

$$E_1 := \{\text{endurance, beauty}\} \text{ and } E_2 := \{\text{endurance, warmth}\}$$

with $\widetilde{\mathrm{G}}_1[\text{endurance}], \widetilde{\mathrm{G}}_1[\text{beauty}], \widetilde{\mathrm{G}}_2[\text{endurance}]$, and $\widetilde{\mathrm{G}}_2[\text{warmth}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{G}}_1$	black	cyan	blue	green
endurance	1	0.5	0.7	0.5
beauty	0.4	0.3	0.2	0.2

$\widetilde{\mathrm{G}}_2$	black	cyan	blue	green
endurance	1	0.6	0.5	0.5
warmth	0.9	0.4	0.5	0.4

Then $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_i}$-filters of A. Since endurance $\in E_1 \cap E_2$, we have

$$(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]})(\text{green}) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} =$$
$$\min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]})(\text{cyan}), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]})(\text{blue})\} =$$
$$\min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]})(\text{blue} \cdot \text{green}), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]})(\text{blue})\}.$$

Thus $\widetilde{\mathrm{G}}_1[\text{endurance}] \cup \widetilde{\mathrm{G}}_2[\text{endurance}]$ is not a fuzzy $\mathrm{UP_i}$-filter of A, that is, $(\widetilde{\mathrm{G}}_1, E_1) \cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a endurance-fuzzy soft $\mathrm{UP_i}$-filter of A. Hence, $(\widetilde{\mathrm{G}}_1, E_1) \cup$

$(\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft UP$_\mathrm{i}$-filter of A. Moreover, $(\widetilde{\mathrm{G}}_1, E_1) \Cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft UP$_\mathrm{i}$-filter of A.

Definition 4.4.58 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft UP$_\mathrm{s}$-ideal* based on $e \in E$ (we shortly call an *e-fuzzy soft UP$_\mathrm{s}$-ideal*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy UP$_\mathrm{s}$-ideal of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft UP$_\mathrm{s}$-ideal of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft UP$_\mathrm{s}$-ideal* of A.

In the next theorem and corollary, we give necessary condition for fuzzy soft UP$_\mathrm{s}$-ideals of f-UP-semigroups.

Theorem 4.4.59 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.8) *and* (3.0.14), *then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{s}$-ideal of A.*

Proof. It is straightforward by Proposition 4.3.16 and Lemma 3.0.36 (1). □

Corollary 4.4.60 *Let A be an f-UP-semigroup satisfying the condition* (4.3.10). *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.9) *and* (3.0.14), *then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{s}$-ideal of A.*

Proof. It is straightforward by Theorems 4.4.59 and 4.3.19. □

Theorem 4.4.61 *Every e-fuzzy soft UP$_\mathrm{s}$-ideal of A is an e-fuzzy soft UP$_\mathrm{s}$-filter. Moreover, every fuzzy soft UP$_\mathrm{s}$-ideal of A is a fuzzy soft UP$_\mathrm{s}$-filter.*

The following example shows that the converse of Theorem 4.4.61 is not true.

Example 4.4.62 By Cayley tables in Example 4.4.18, we know that $A = (A, \cdot, *, \mathrm{pop})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over A where

$$E := \{\text{sorrow, relaxation, enjoyment}\}$$

with $\widetilde{\mathrm{F}}[\mathrm{sorrow}], \widetilde{\mathrm{F}}[\mathrm{modernity}]$, and $\widetilde{\mathrm{F}}[\mathrm{enjoyment}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}$	pop	rock	disco	classic
sorrow	0.6	0.2	0.1	0.1
modernity	1	0.5	0.5	0.5
enjoyment	0.7	0.5	0.2	0.2

Then $(\widetilde{\mathrm{F}}, E)$ is a sorrow-fuzzy soft $\mathrm{UP_s}$-filter of A but $\widetilde{\mathrm{F}}[\mathrm{sorrow}]$ is not a fuzzy $\mathrm{UP_s}$-ideal of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{disco}\cdot\mathrm{classic}) = \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{disco}) = 0.1 \ngeq 0.2 = \min\{0.6, 0.2\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{pop}), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{rock})\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{disco}\cdot(\mathrm{rock}\cdot\mathrm{classic})), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{sorrow}]}(\mathrm{rock})\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a sorrow-fuzzy soft $\mathrm{UP_s}$-ideal of A.

In the next theorem, we give necessary condition for fuzzy soft $\mathrm{UP_s}$-filters as fuzzy soft $\mathrm{UP_s}$-ideals of f-UP-semigroups.

Theorem 4.4.63 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP_{s}-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.11), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP_{s}-ideal of A.*

Proof. It is straightforward by Theorem 4.3.24. □

The proof of the following theorem can be verified easily.

Theorem 4.4.64 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP_{s}-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft UP_{s}-ideal of A.*

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.65 *The extended intersection of two fuzzy soft UP_{s}-ideals of A is also a fuzzy soft UP_{s}-ideal. Moreover, the intersection of two fuzzy soft UP_{s}-ideals of A is also a fuzzy soft UP_{s}-ideal.*

Theorem 4.4.66 *The union of two fuzzy soft UP_{s}-ideals of A is also a fuzzy soft UP_{s}-ideal if sets of statistics of two fuzzy soft UP_{s}-ideals are disjoint.*

The following example shows that Theorem 4.4.66 is not valid if sets of statistics of two fuzzy soft $\mathrm{UP_s}$-ideals are not disjoint.

Example 4.4.67 In Example 4.4.10, we have $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_s}$-ideals of A. Since price $\in E_1 \cap E_2$, we have

$$(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(6 * 5) = (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(7) = 0.7 \ngeq 0.8 = \min\{0.9, 0.8\} = \min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(6), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]})(5)\}.$$

Thus $\widetilde{\mathrm{G}}_1[\mathrm{price}]\cup\widetilde{\mathrm{G}}_2[\mathrm{price}]$ is not a fuzzy $\mathrm{UP_s}$-ideal of A, that is, $(\widetilde{\mathrm{G}}_1, E_1)\cup(\widetilde{\mathrm{G}}_2, E_2)$ is not a price-fuzzy soft $\mathrm{UP_s}$-ideal of A. Hence, $(\widetilde{\mathrm{G}}_1, E_1) \cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_s}$-ideal of A. Moreover, $(\widetilde{\mathrm{G}}_1, E_1) \Cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_s}$-ideal of A.

Definition 4.4.68 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft UP_{i}-ideal* based on $e \in E$ (we shortly call an *e-fuzzy soft UP_{i}-ideal*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy $\mathrm{UP_i}$-ideal of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft $\mathrm{UP_i}$-ideal of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft UP_{i}-ideal* of A.

In the next theorem and corollary, we give necessary condition for fuzzy soft $\mathrm{UP_i}$-ideals of f-UP-semigroups.

Theorem 4.4.69 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.8) *and* (3.0.15), *then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP_{i}-ideal of A.*

Proof. It is straightforward by Proposition 4.3.16 and Lemma 3.0.36 (2). □

Corollary 4.4.70 *Let A be an f-UP-semigroup satisfying the condition* (4.3.10). *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.9) *and* (3.0.15), *then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_{\mathrm{i}}$-ideal of A.*

Proof. It is straightforward by Theorems 4.4.69 and 4.3.19. □

Theorem 4.4.71 *Every e-fuzzy soft UP$_{\mathrm{i}}$-ideal of A is an e-fuzzy soft UP$_{\mathrm{s}}$-ideal. Moreover, every fuzzy soft UP$_{\mathrm{i}}$-ideal of A is a fuzzy soft UP$_{\mathrm{s}}$-ideal.*

Theorem 4.4.72 *Every e-fuzzy soft UP$_{\mathrm{i}}$-ideal of A is an e-fuzzy soft UP$_{\mathrm{i}}$-filter. Moreover, every fuzzy soft UP$_{\mathrm{i}}$-ideal of A is a fuzzy soft UP$_{\mathrm{i}}$-filter.*

The following two examples show that the converse of Theorems 4.4.71 and 4.4.72 is not true.

Example 4.4.73 In Example 4.4.7, we know that $(\widetilde{\mathrm{F}}, E)$ is a price-fuzzy soft UP$_{\mathrm{s}}$-ideal of A but $\widetilde{\mathrm{F}}$[price] is not a fuzzy UP$_{\mathrm{i}}$-ideal of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(5 * 6) = \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(7) = 0.3 \not\geq 0.7 = \max\{0.1, 0.7\} = \\ \max\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(5), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{price}]}(6)\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a price-fuzzy soft UP$_{\mathrm{i}}$-ideal of A.

Example 4.4.74 In Example 4.4.62, we know that $(\widetilde{\mathrm{F}}, E)$ is a enjoyment-fuzzy soft UP$_{\mathrm{i}}$-filter of A but $\widetilde{\mathrm{F}}$[enjoyment] is not a fuzzy UP$_{\mathrm{i}}$-ideal of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{disco} \cdot \mathrm{classic}) = \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{disco}) = 0.2 \not\geq 0.5 = \min\{0.7, 0.5\} = \\ \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{pop}), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{rock})\} = \\ \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{disco} \cdot (\mathrm{rock} \cdot \mathrm{classic})), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{enjoyment}]}(\mathrm{rock})\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a enjoyment-fuzzy soft $\mathrm{UP_i}$-ideal of A.

In the next theorem, we give necessary condition for fuzzy soft $\mathrm{UP_i}$-filters as fuzzy soft $\mathrm{UP_i}$-ideals of f-UP-semigroups.

Theorem 4.4.75 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_{i}*-filter of* A *such that for all* $e \in E$*, a fuzzy set* $\widetilde{\mathrm{F}}[e]$ *in* A *satisfies the condition* (4.3.11)*, then* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_{i}*-ideal of* A*.*

Proof. It is straightforward by Theorem 4.3.24. □

The proof of the following theorem can be verified easily.

Theorem 4.4.76 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft* UP_{i}*-ideal of* A *and* $\emptyset \neq E^* \subseteq E$*, then* $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ *is a fuzzy soft* UP_{i}*-ideal of* A*.*

The following two theorems can be deduced in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.77 *The extended intersection of two fuzzy soft* UP_{i}*-ideals of* A *is also a fuzzy soft* UP_{i}*-ideal. Moreover, the intersection of two fuzzy soft* UP_{i}*-ideals of* A *is also a fuzzy soft* UP_{i}*-ideal.*

Theorem 4.4.78 *The union of two fuzzy soft* UP_{i}*-ideals of* A *is also a fuzzy soft* UP_{i}*-ideal if sets of statistics of two fuzzy soft* UP_{i}*-ideals are disjoint.*

The following example shows that the converse of Theorem 4.4.78 is not true.

Example 4.4.79 In Example 4.4.57, we have $(\widetilde{\mathrm{G}}_1, E_1)$ and $(\widetilde{\mathrm{G}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_i}$-ideals of A. Since endurance $\in E_1 \cap E_2$, we have

$$(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{black}\cdot\mathrm{green}) = (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{green}) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{cyan}), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{blue})\} = \min\{(\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{black}\cdot(\mathrm{blue}\cdot\mathrm{green})), (\mathrm{f}_{\widetilde{\mathrm{G}}_1[\mathrm{endurance}]\cup\widetilde{\mathrm{G}}_2[\mathrm{endurance}]})(\mathrm{blue})\}.$$

Thus $\widetilde{\mathrm{G}}_1[\mathrm{endurance}] \cup \widetilde{\mathrm{G}}_2[\mathrm{endurance}]$ is not a fuzzy $\mathrm{UP_i}$-ideal of A, that is, $(\widetilde{\mathrm{G}}_1, E_1)\cup(\widetilde{\mathrm{G}}_2, E_2)$ is not a endurance-fuzzy soft $\mathrm{UP_i}$-ideal of A. Hence, $(\widetilde{\mathrm{G}}_1, E_1)\cup(\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_i}$-ideal of A. Moreover, $(\widetilde{\mathrm{G}}_1, E_1) \Cup (\widetilde{\mathrm{G}}_2, E_2)$ is not a fuzzy soft $\mathrm{UP_i}$-ideal of A.

Definition 4.4.80 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft strongly UP_{s}-ideal* based on $e \in E$ (we shortly call an *e-fuzzy soft strongly UP_{s}-ideal*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy strongly $\mathrm{UP_s}$-ideal of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft strongly $\mathrm{UP_s}$-ideal of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft strongly UP_{s}-ideal* of A.

Definition 4.4.81 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *constant fuzzy soft set* based on $e \in E$ (we shortly call an *e-constant fuzzy soft set*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is constant. If $(\widetilde{\mathrm{F}}, E)$ is an e-constant fuzzy soft set over A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *constant fuzzy soft set* over A.

Theorem 4.4.82 *Every e-fuzzy soft strongly UP_{s}-ideal of A is an e-fuzzy soft UP_{s}-ideal. Moreover, every fuzzy soft strongly UP_{s}-ideal of A is a fuzzy soft UP_{s}-ideal.*

Theorem 4.4.83 *e-fuzzy soft strongly UP_{s}-ideals and e-constant fuzzy soft sets coincide in A. Moreover, fuzzy soft strongly UP_{s}-ideals and constant fuzzy soft sets coincide in A.*

In the next theorem, we give necessary condition for fuzzy soft strongly $\mathrm{UP_s}$-ideals of f-UP-semigroups.

Theorem 4.4.84 *If* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft set over* A *such that for all* $e \in E$*, a fuzzy set* $\widetilde{\mathrm{F}}[e]$ *in* A *satisfies the conditions* (4.3.12) *(or* (4.3.13) *or* (4.3.14)*) and* (3.0.14)*, then* $(\widetilde{\mathrm{F}}, E)$ *is a fuzzy soft strongly* UP_{s}*-ideal of* A*.*

Proof. It is straightforward by Propositions 4.3.25 (or 4.3.27 or 4.3.29) and Lemma 3.0.36 (1). □

The following example shows that the converse of Theorem 4.4.82 is not true.

Example 4.4.85 Let A be a set of four brands of a pick-up truck, that is,

$$A = \{\text{Toyota Hilux(TH), Mitsubishi Triton(MT), Ford Ranger(FR),}$$
$$\text{Isuzu D-Max(ID)}\}.$$

Define two binary operations $\cdot$ and $*$ on A as the following Cayley tables:

$\cdot$	MT	FR	ID	TH
MT	MT	FR	ID	TH
FR	MT	MT	ID	TH
ID	MT	FR	MT	TH
TH	MT	FR	ID	MT

$*$	MT	FR	ID	TH
MT	MT	MT	MT	MT
FR	MT	FR	MT	MT
ID	MT	MT	ID	MT
TH	MT	TH	MT	MT

Then $A = (A, \cdot, *, \text{Mitsubishi Triton})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}, E)$ be a fuzzy soft set over A where

$$E := \{\text{displacement, horse power, torque}\}$$

with $\widetilde{\mathrm{F}}[\text{displacement}], \widetilde{\mathrm{F}}[\text{horse power}]$, and $\widetilde{\mathrm{F}}[\text{torque}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}$	MT	FR	ID	TH
displacement	1	0.6	0.4	0.7
horse power	0.9	0.6	0.5	0.5
torque	0.9	0.7	0.6	0.5

Then $(\widetilde{\mathrm{F}}, E)$ is a torque-fuzzy soft $\mathrm{UP_s}$-ideal of A but $\widetilde{\mathrm{F}}[\text{torque}]$ is not a fuzzy strongly $\mathrm{UP_s}$-ideal of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\text{torque}]}(\mathrm{ID}) = 0.6 \ngeq 0.7 = \min\{0.9, 0.7\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{torque}]}(\mathrm{MT}), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{torque}]}(\mathrm{FR})\} = \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\text{torque}]}((\mathrm{ID} \cdot \mathrm{FR}) \cdot (\mathrm{ID} \cdot \mathrm{ID})), \mathrm{f}_{\widetilde{\mathrm{F}}[\text{torque}]}(\mathrm{FR})\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a torque-fuzzy soft strongly $\mathrm{UP_s}$-ideal of A.

The proof of the following theorem can be verified easily.

Theorem 4.4.86 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly UP_{s}-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft strongly UP_{s}-ideal of A.*

By using Theorem 4.2.32, we can obtain the following two theorems in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.87 *The extended intersection of two fuzzy soft strongly UP_{s}-ideals of A is also a fuzzy soft strongly UP_{s}-ideal. Moreover, the intersection of two fuzzy soft strongly UP_{s}-ideals of A is also a fuzzy soft strongly UP_{s}-ideal.*

Theorem 4.4.88 *The union of two fuzzy soft strongly UP_{s}-ideals is also a fuzzy soft strongly UP_{s}-ideal. Moreover, the restricted union of two fuzzy soft strongly UP_{s}-ideals of A is also a fuzzy soft strongly UP_{s}-ideal.*

Definition 4.4.89 A fuzzy soft set $(\widetilde{\mathrm{F}}, E)$ over A is called a *fuzzy soft strongly UP_{i}-ideal* based on $e \in E$ (we shortly call an *e-fuzzy soft strongly UP_{i}-ideal*) of A if a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A is a fuzzy strongly $\mathrm{UP_i}$-ideal of A. If $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft strongly $\mathrm{UP_i}$-ideal of A for all $e \in E$, we say that $(\widetilde{\mathrm{F}}, E)$ is a *fuzzy soft strongly UP_i-ideal* of A.

Theorem 4.4.90 *Every e-fuzzy soft strongly UP_i-ideal of A is an e-fuzzy soft UP_i-ideal. Moreover, every fuzzy soft strongly UP_i-ideal of A is a fuzzy soft UP_i-ideal.*

Theorem 4.4.91 *e-fuzzy soft strongly UP_i-ideals and e-constant fuzzy soft sets coincide in A. Moreover, fuzzy soft strongly UP_i-ideals and constant fuzzy soft sets coincide in A.*

Corollary 4.4.92 *e-fuzzy soft strongly UP_s-ideals, e-fuzzy soft strongly UP_i-ideals, and e-constant fuzzy soft sets coincide in A. Moreover, fuzzy soft strongly UP_s-ideals, fuzzy soft strongly UP_i-ideals and constant fuzzy soft sets coincide in A.*

Proof. It is straightforward by Theorems 4.4.83 and 4.4.91. □

In the next theorem, we give necessary condition for fuzzy soft strongly $\mathrm{UP_i}$-ideals of f-UP-semigroups.

Theorem 4.4.93 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions* (4.3.12) *(or* (4.3.13) *or* (4.3.14)*) and* (3.0.15)*, then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly UP_i-ideal of A.*

Proof. It is straightforward by Proposition 4.3.25 (or 4.3.27 or 4.3.29) and Lemma 3.0.36 (2). □

The following example shows that the converse of Theorem 4.4.90 is not true.

Example 4.4.94 In Example 4.4.85, we know that $(\widetilde{\mathrm{F}}, E)$ is a displacement-fuzzy soft $\mathrm{UP_i}$-ideal of A but $\widetilde{\mathrm{F}}$[displacement] is not a fuzzy strongly $\mathrm{UP_i}$-ideal of A. Indeed,

$$\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{displacement}]}(\mathrm{ID}) = 0.4 \not\geq 0.6 = \min\{1, 0.6\} = \\ \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{displacement}]}(\mathrm{MT}), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{displacement}]}(\mathrm{FR})\} = \\ \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{displacement}]}((\mathrm{ID}\cdot\mathrm{FR})\cdot(\mathrm{ID}\cdot\mathrm{ID})), \mathrm{f}_{\widetilde{\mathrm{F}}[\mathrm{displacement}]}(\mathrm{FR})\}.$$

Hence, $(\widetilde{\mathrm{F}}, E)$ is not a displacement-fuzzy soft strongly $\mathrm{UP_i}$-ideal of A.

The proof of the following theorem can be verified easily.

Theorem 4.4.95 *If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly $\mathrm{UP_i}$-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft strongly $\mathrm{UP_i}$-ideal of A.*

By using Theorem 4.2.33, we can obtain the following two theorems in the same way as Theorems 4.4.8 and 4.4.9.

Theorem 4.4.96 *The extended intersection of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal. Moreover, the intersection of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal.*

Theorem 4.4.97 *The union of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal. Moreover, the restricted union of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal.*

Then, we get the diagram of generalization of fuzzy soft sets over fully UP-semigroups as shown in Figure 4.4 below.

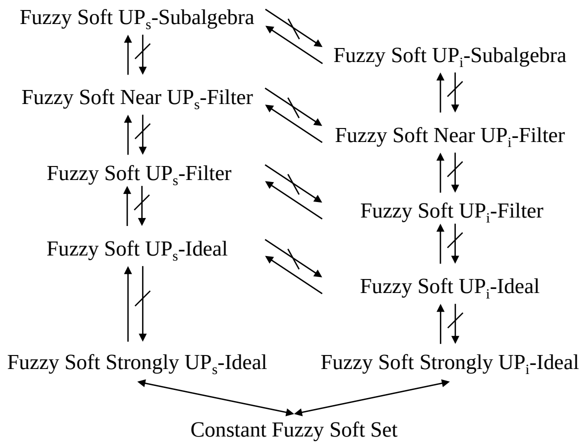

Figure 3: Fuzzy soft sets over fully UP-semigroups

4.5 Properties of operations for fuzzy soft sets over fully UP-semigroups

From now on, we shall let A be an f-UP-semigroup $A = (A, \cdot, *, 0)$ and P be a set of parameters. Let $\mathcal{F}(A)$ denotes the set of all fuzzy sets in A. A subset E of P is called a *set of statistics*.

Definition 4.5.1 [24] Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft sets over a common universe U. The *OR* of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \vee (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \times E_2$ and

(ii) $\widetilde{\mathrm{H}}[e_1, e_2] = \widetilde{\mathrm{F}}[e_1] \cup \widetilde{\mathrm{G}}[e_2]$ for all $(e_1, e_2) \in E$.

Definition 4.5.2 [24] Let $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ be two fuzzy soft sets over a common universe U. The *AND* of $(\widetilde{\mathrm{F}}, E_1)$ and $(\widetilde{\mathrm{G}}, E_2)$ is defined to be the fuzzy soft set $(\widetilde{\mathrm{F}}, E_1) \wedge (\widetilde{\mathrm{G}}, E_2) = (\widetilde{\mathrm{H}}, E)$ satisfying the following conditions:

(i) $E = E_1 \times E_2$ and

(ii) $\widetilde{\mathrm{H}}[e_1, e_2] = \widetilde{\mathrm{F}}[e_1] \cap \widetilde{\mathrm{G}}[e_2]$ for all $(e_1, e_2) \in E$.

We will introduce the notions of the restricted union, the union, the intersection, the extended intersection, the AND, and the OR of any fuzzy soft sets and apply to f-UP-semigroups.

Definition 4.5.3 Let $\{(\widetilde{\mathrm{F}}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The *restricted union* of $(\widetilde{\mathrm{F}}_i, E_i)$ is defined to be the fuzzy soft set $\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ satisfying the following conditions:

(i) $E = \bigcap_{i\in I} E_i \neq \emptyset$ and

(ii) $\widetilde{\mathrm{F}}[e] = \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e]$ for all $e \in E$.

Theorem 4.5.4 *The restricted union of family of fuzzy soft near UP*$_\mathrm{i}$*-filters of A is also a fuzzy soft near UP*$_\mathrm{i}$*-filter.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft near UP$_\mathrm{i}$-filters of A for all $i \in I$. Assume that $\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ be the restricted union of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E = \bigcap_{i\in I} E_i \neq \emptyset$. Let $e \in E$. By Theorem 4.2.10, we have $\widetilde{\mathrm{F}}[e] = \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e]$ is a fuzzy near UP$_\mathrm{i}$-filter of A. Therefore, $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft near UP$_\mathrm{i}$-filter of A. But since e is an arbitrary statistic of E, we have $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near UP$_\mathrm{i}$-filter of A. □

In the same way as Theorem 4.5.4, we can use Theorems 4.2.32 (resp., 4.2.33) to prove that the restricted union of family of fuzzy soft strongly UP$_\mathrm{s}$-ideals (resp., fuzzy soft strongly UP$_\mathrm{i}$-ideals) of A is also a fuzzy soft strongly UP$_\mathrm{s}$-ideal (resp., fuzzy soft strongly UP$_\mathrm{i}$-ideal).

Definition 4.5.5 Let $\{(\widetilde{\mathrm{F}}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The *union* of $(\widetilde{\mathrm{F}}_i, E_i)$ is defined to be the fuzzy soft set $\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ satisfying the following conditions:

(i) $E = \bigcup_{i \in I} E_i$ and

(ii) $\widetilde{\mathrm{F}}[e] = \bigcup_{j \in J} \widetilde{\mathrm{F}}_j[e]$ for all $e \in E$ with $e \in \bigcap_{j \in J} E_j - \bigcup_{k \in I-J} E_k$ where $\emptyset \neq J \subseteq I$.

Theorem 4.5.6 *The union of family of fuzzy soft near UP*$_{\mathrm{i}}$*-filters of A is also a fuzzy soft near UP*$_{\mathrm{i}}$*-filter.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft near UP$_{\mathrm{i}}$-filters of A for all $i \in I$. Assume that $\bigcap_{i \in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ be the union of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E = \bigcup_{i \in I} E_i$. Let $e \in E$.

Case 1: $|J| = |I|$. By Theorem 4.5.4, we have $\widetilde{\mathrm{F}}[e] = \bigcap_{i \in I} \widetilde{\mathrm{F}}_i[e]$ is a fuzzy near UP$_{\mathrm{i}}$-filter of A.

Case 2: $|J| = 1$, that is, J is a singleton set. Then $\widetilde{\mathrm{F}}[e] = \bigcap_{j \in \{j\}} \widetilde{\mathrm{F}}_j[e] = \widetilde{\mathrm{F}}_j[e]$ is a fuzzy near UP$_{\mathrm{i}}$-filter of A.

Case 3: $1 < |J| < |I|$. Then $\widetilde{\mathrm{F}}[e] = \bigcap_{j \in J} \widetilde{\mathrm{F}}_j[e]$. Since $e \in E_j$ for all $j \in J$ and $e \notin E_k$ for some $k \in I - J$ and by same Case 1, we have $\widetilde{\mathrm{F}}[e]$ is a fuzzy near UP$_{\mathrm{i}}$-filter of A.

Therefore, $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft near UP$_{\mathrm{i}}$-filter of A. But since e is an arbitrary statistic of E, we have $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near UP$_{\mathrm{i}}$-filter of A. □

In the same way as Theorem 4.5.6, we can prove that the union of family of fuzzy soft strongly UP$_{\mathrm{s}}$-ideals (resp., fuzzy soft strongly UP$_{\mathrm{i}}$-ideals) of A is also a fuzzy soft strongly UP$_{\mathrm{s}}$-ideal (resp., fuzzy soft strongly UP$_{\mathrm{i}}$-ideal).

In section 4.4, we show that the union of two fuzzy soft UP$_{\mathrm{s}}$-subalgebras (resp., fuzzy soft UP$_{\mathrm{i}}$-subalgebras, fuzzy soft near UP$_{\mathrm{s}}$-filters, fuzzy soft UP$_{\mathrm{s}}$-filters, fuzzy soft UP$_{\mathrm{i}}$-filters, fuzzy soft UP$_{\mathrm{s}}$-ideals, fuzzy soft UP$_{\mathrm{i}}$-ideals) of A is not fuzzy soft UP$_{\mathrm{s}}$-subalgebra (resp., fuzzy soft UP$_{\mathrm{i}}$-subalgebra, fuzzy soft near

UP$_\mathrm{s}$-filter, fuzzy soft UP$_\mathrm{s}$-filter, fuzzy soft UP$_\mathrm{i}$-filter, fuzzy soft UP$_\mathrm{s}$-ideal, fuzzy soft UP$_\mathrm{i}$-ideal).

Definition 4.5.7 Let $\{(\widetilde{\mathrm{F}}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The *intersection* of $(\widetilde{\mathrm{F}}_i, E_i)$ is defined to be the fuzzy soft set $\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ satisfying the following conditions:

(i) $E = \bigcap_{i\in I} E_i \neq \emptyset$ and

(ii) $\widetilde{\mathrm{F}}[e] = \bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e]$ for all $e \in E$.

Theorem 4.5.8 *The intersection of family of fuzzy soft UP$_\mathrm{s}$-subalgebras of A is also a fuzzy soft UP$_\mathrm{s}$-subalgebra.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft UP$_\mathrm{s}$-subalgebras of A for all $i \in I$. Assume that $\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ is the intersection of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E = \bigcap_{i\in I} E_i \neq \emptyset$. Let $e \in E$. By Theorem 4.2.2, we have $\widetilde{\mathrm{F}}[e] = \bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e]$ is a fuzzy UP$_\mathrm{s}$-subalgebra of A. Therefore, $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft UP$_\mathrm{s}$-subalgebra of A. But since e is an arbitrary statistic of E, we have $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft UP$_\mathrm{s}$-subalgebra of A. □

In the same way as Theorem 4.5.8, we can use Theorems 4.2.4 (resp., 4.2.7, 4.2.9, 4.2.15, 4.2.17, 4.2.24, 4.2.26, 4.2.32, 4.2.33) to prove that the intersection of family of fuzzy soft UP$_\mathrm{i}$-subalgebras (resp., fuzzy soft near UP$_\mathrm{s}$-filters, fuzzy soft near UP$_\mathrm{i}$-filters, fuzzy soft UP$_\mathrm{s}$-filters, fuzzy soft UP$_\mathrm{i}$-filters, fuzzy soft UP$_\mathrm{s}$-ideals, fuzzy soft UP$_\mathrm{i}$-ideals, fuzzy soft strongly UP$_\mathrm{s}$-ideals, fuzzy soft strongly UP$_\mathrm{i}$-ideals) of A is also a fuzzy soft UP$_\mathrm{i}$-subalgebra (resp., fuzzy soft near UP$_\mathrm{s}$-filter, fuzzy soft near UP$_\mathrm{i}$-filter, fuzzy soft UP$_\mathrm{s}$-filter, fuzzy soft UP$_\mathrm{i}$-filter, fuzzy soft UP$_\mathrm{s}$-ideal, fuzzy soft UP$_\mathrm{i}$-ideal, fuzzy soft strongly UP$_\mathrm{s}$-ideal, fuzzy soft strongly UP$_\mathrm{i}$-ideal).

Definition 4.5.9 Let $\{(\widetilde{\mathrm{F}}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The *extended intersection* of $(\widetilde{\mathrm{F}}_i, E_i)$ is defined to be the fuzzy soft set $\bigcap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ satisfying the following conditions:

(i) $E = \bigcup_{i\in I} E_i$ and

(ii) $\widetilde{\mathrm{F}}[e] = \bigcap_{j\in J} \widetilde{\mathrm{F}}_j[e]$ for all $e \in E$ with $e \in \bigcap_{j\in J} E_j - \bigcup_{k\in I-J} E_k$ where $\emptyset \neq J \subseteq I$.

Theorem 4.5.10 *The extended intersection of family of fuzzy soft* UP_{s}*-subalgebras of* A *is also a fuzzy soft* UP_{s}*-subalgebra.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft $\mathrm{UP_s}$-subalgebras of A for all $i \in I$. Assume that $\bigcap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ is the extended intersection of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E = \bigcup_{i\in I} E_i$. Let $e \in E$.

Case 1: $|J| = |I|$. By Theorem 4.5.8, we have $\widetilde{\mathrm{F}}[e] = \bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e]$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Case 2: $|J| = 1$, that is, J is a singleton set. Then $\widetilde{\mathrm{F}}[e] = \bigcap_{j\in\{j\}} \widetilde{\mathrm{F}}_j[e] = \widetilde{\mathrm{F}}_j[e]$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Case 3: $1 < |J| < |I|$. Then $\widetilde{\mathrm{F}}[e] = \bigcap_{j\in J} \widetilde{\mathrm{F}}_j[e]$. Since $e \in E_j$ for all $j \in J$ and $e \notin E_k$ for some $k \in I - J$ and by same Case 1, we have $\widetilde{\mathrm{F}}[e]$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Therefore, $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra of A. But since e is an arbitrary statistic of E, we have $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A. □

In the same way as Theorem 4.5.10, we can prove that the extended intersection of family of fuzzy soft $\mathrm{UP_i}$-subalgebras (resp., fuzzy soft near $\mathrm{UP_s}$-

filters, fuzzy soft near $\mathrm{UP_i}$-filters, fuzzy soft $\mathrm{UP_s}$-filters, fuzzy soft $\mathrm{UP_i}$-filters, fuzzy soft $\mathrm{UP_s}$-ideals, fuzzy soft $\mathrm{UP_i}$-ideals, fuzzy soft strongly $\mathrm{UP_s}$-ideals, fuzzy soft strongly $\mathrm{UP_i}$-ideals) of A is also a fuzzy soft $\mathrm{UP_i}$-subalgebra (resp., fuzzy soft near $\mathrm{UP_s}$-filter, fuzzy soft near $\mathrm{UP_i}$-filter, fuzzy soft $\mathrm{UP_s}$-filter, fuzzy soft $\mathrm{UP_i}$-filter, fuzzy soft $\mathrm{UP_s}$-ideal, fuzzy soft $\mathrm{UP_i}$-ideal, fuzzy soft strongly $\mathrm{UP_s}$-ideal, fuzzy soft strongly $\mathrm{UP_i}$-ideal).

Definition 4.5.11 Let $\{(\widetilde{\mathrm{F}}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The AND of $(\widetilde{\mathrm{F}}_i, E_i)$ is defined to be the fuzzy soft set $\bigwedge_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ satisfying the following conditions:

(i) $E = \prod_{i\in I} E_i$ and

(ii) $\widetilde{\mathrm{F}}[(e_i)_{i\in I}] = \bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e_i]$ for all $(e_i)_{i\in I} \in E$.

Theorem 4.5.12 *The* AND *of family of fuzzy soft* $\mathrm{UP_s}$*-subalgebras of* A *is also a fuzzy soft* $\mathrm{UP_s}$*-subalgebra.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft $\mathrm{UP_s}$-subalgebras of A for all $i \in I$. By means of Definition 4.5.11, we assume that $\bigwedge_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ such that $E = \prod_{i\in I} E_i$ and $\widetilde{\mathrm{F}}[(e_i)_{i\in I}] = \bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e_i]$ for all $(e_i)_{i\in I} \in E$. Assume that $e = (e_i)_{i\in I} \in E$ and let $x, y \in A$. Then

$$
\begin{aligned}
\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x \cdot y) &= \mathrm{f}_{\bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e_i]}(x \cdot y) \\
&= \inf\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x \cdot y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x), \mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e_i]}(x), \mathrm{f}_{\bigcap_{i\in I} \widetilde{\mathrm{F}}_i[e_i]}(y)\} \\
&= \min\{\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x), \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(y)\}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
f_{\widetilde{F}[e]}(x * y) &= f_{\bigcap_{i\in I}\widetilde{F}_i[e_i]}(x * y) \\
&= \inf\{f_{\widetilde{F}_i[e_i]}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{f_{\widetilde{F}_i[e_i]}(x), f_{\widetilde{F}_i[e_i]}(y)\}\}_{i\in I} \\
&= \min\{\inf\{f_{\widetilde{F}_i[e_i]}(x)\}_{i\in I}, \inf\{f_{\widetilde{F}_i[e_i]}(y)\}_{i\in I}\} \\
&= \min\{f_{\bigcap_{i\in I}\widetilde{F}_i[e_i]}(x), f_{\bigcap_{i\in I}\widetilde{F}_i[e_i]}(y)\} \\
&= \min\{f_{\widetilde{F}[e]}(x), f_{\widetilde{F}[e]}(y)\}.
\end{aligned}
$$

Therefore, $\widetilde{F}[e]$ is a fuzzy UP_s-subalgebra of A, that is, $(\widetilde{F}, E)$ is an e-fuzzy soft UP_s-subalgebra of A. But since e is an arbitrary statistic of E, we have $(\widetilde{F}, E)$ is a fuzzy soft UP_s-subalgebra of A. □

In the same way as Theorem 4.5.12, we can prove that the AND of family of fuzzy soft UP_i-subalgebras (resp., fuzzy soft near UP_s-filters, fuzzy soft near UP_i-filters, fuzzy soft UP_s-filters, fuzzy soft UP_i-filters, fuzzy soft UP_s-ideals, fuzzy soft UP_i-ideals, fuzzy soft strongly UP_s-ideals, fuzzy soft strongly UP_i-ideals) of A is also a fuzzy soft UP_i-subalgebra (resp., fuzzy soft near UP_s-filter, fuzzy soft near UP_i-filter, fuzzy soft UP_s-filter, fuzzy soft UP_i-filter, fuzzy soft UP_s-ideal, fuzzy soft UP_i-ideal, fuzzy soft strongly UP_s-ideal, fuzzy soft strongly UP_i-ideal).

Definition 4.5.13 Let $\{(\widetilde{F}_i, E_i) \mid i \in I\}$ be a nonempty family of fuzzy soft sets over a common universe U where I is an arbitrary index set. The OR of $(\widetilde{F}_i, E_i)$ is defined to be the fuzzy soft set $\bigvee_{i\in I}(\widetilde{F}_i, E_i) = (\widetilde{F}, E)$ satisfying the following conditions:

(i) $E = \prod_{i\in I} E_i$ and

(ii) $\widetilde{F}[(e_i)_{i\in I}] = \bigcup_{i\in I} \widetilde{F}_i[e_i]$ for all $(e_i)_{i\in I} \in E$.

Theorem 4.5.14 *The* OR *of family of fuzzy soft near* UP_{i}*-filters of* A *is also a fuzzy soft near* UP_{i}*-filter.*

Proof. Let $(\widetilde{\mathrm{F}}_i, E_i)$ be a fuzzy soft near $\mathrm{UP_i}$-filters of A for all $i \in I$. By means of Definition 4.5.13, we assume that $\bigvee_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{F}}, E)$ such that $E = \prod_{i\in I} E_i$ and $\widetilde{\mathrm{F}}[(e_i)_{i\in I}] = \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e_i]$ for all $(e_i)_{i\in I} \in E$. Assume that $e = (e_i)_{i\in I} \in E$ and let $x, y \in A$. Then

$$\begin{aligned}
\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(0) &= \mathrm{f}_{\bigcup_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(0) \\
&= \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(0)\}_{i\in I} \\
&\geq \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigcup_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(x) \\
&= \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x), \\
\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x \cdot y) &= \mathrm{f}_{\bigcup_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(x \cdot y) \\
&= \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x \cdot y)\}_{i\in I} \\
&\geq \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(y)\}_{i\in I} \\
&= \mathrm{f}_{\bigcup_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(y) \\
&= \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(y), \text{ and} \\
\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x * y) &= \mathrm{f}_{\bigcup_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(x * y) \\
&= \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x * y)\}_{i\in I} \\
&\geq \sup\{\max\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x), \mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(y)\}\}_{i\in I} \\
&= \max\{\sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(x)\}_{i\in I}, \sup\{\mathrm{f}_{\widetilde{\mathrm{F}}_i[e_i]}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigcap_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(x), \mathrm{f}_{\bigcap_{i\in I}\widetilde{\mathrm{F}}_i[e_i]}(y)\} \\
&= \max\{\mathrm{f}_{\widetilde{\mathrm{F}}[e]}(x), \mathrm{f}_{\widetilde{\mathrm{F}}[e]}(y)\}.
\end{aligned}$$

Therefore, $\widetilde{\mathrm{F}}[e]$ is a fuzzy near $\mathrm{UP_i}$-filter of A, that is, $(\widetilde{\mathrm{F}}, E)$ is an e-fuzzy soft near $\mathrm{UP_i}$-filter of A. But since e is an arbitrary statistic of E, we have $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A. □

In the same way as Theorem 4.5.14, we can prove that the OR of family of fuzzy soft strongly $\mathrm{UP_s}$-ideals (resp., fuzzy soft strongly $\mathrm{UP_i}$-ideals) of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal (resp., fuzzy soft strongly $\mathrm{UP_i}$-ideal).

The following example shows that the OR of two fuzzy soft $\mathrm{UP_s}$-subalgebras of A are not fuzzy soft $\mathrm{UP_s}$-subalgebra.

Example 4.5.15 By Cayley tables in Example 4.4.7, we know that $A = (A, \cdot, *, \mathrm{X})$ is an f-UP-semigroup. Let $(\widetilde{\mathrm{F}}_1, E_1)$ and $(\widetilde{\mathrm{F}}_2, E_2)$ be two fuzzy soft sets over A where

$$E_1 := \{\text{price, beauty, specifications}\} \text{ and } E_2 := \{\text{price, stability}\}$$

with $\widetilde{\mathrm{F}}_1[\text{price}], \widetilde{\mathrm{F}}_1[\text{beauty}], \widetilde{\mathrm{F}}_1[\text{specifications}], \widetilde{\mathrm{F}}_2[\text{price}]$, and $\widetilde{\mathrm{F}}_2[\text{stability}]$ are fuzzy sets in A defined as follows:

$\widetilde{\mathrm{F}}_1$	X	7	6	5
price	0.9	0.7	0.9	0.2
beauty	1	0.8	0.3	0.2
specifications	0.6	0.5	0.3	0.4

$\widetilde{\mathrm{F}}_2$	X	7	6	5
price	0.9	0.3	0.2	0.8
stability	0.7	0.2	0.5	0.2

Then $(\widetilde{\mathrm{F}}_1, E_1)$ and $(\widetilde{\mathrm{F}}_2, E_2)$ are two fuzzy soft $\mathrm{UP_s}$-subalgebras of A. Since $(\text{price}, \text{price}) \in E_1 \times E_2$, we have

$$\begin{aligned}(\mathrm{f}_{\widetilde{\mathrm{F}}_1[\text{price}] \cup \widetilde{\mathrm{F}}_2[\text{price}]})(5 * 6) &= (\mathrm{f}_{\widetilde{\mathrm{F}}_1[\text{price}] \cup \widetilde{\mathrm{F}}_2[\text{price}]})(7) \\ &= 0.7 \\ &\ngeq 0.8 \\ &= \min\{0.8, 0.9\}\end{aligned}$$

$$= \min\{(f_{\widetilde{F}_1[\text{price}] \cup \widetilde{F}_2[\text{price}]})(5), (f_{\widetilde{F}_1[\text{price}] \cup \widetilde{F}_2[\text{price}]})(6)\}.$$

Thus $\widetilde{F}_1[\text{price}] \cup \widetilde{F}_2[\text{price}]$ is not a fuzzy UP_s-subalgebra of A, that is, $(\widetilde{F}_1, E_1) \cup (\widetilde{F}_2, E_2)$ is not a (price, price)-fuzzy soft UP_s-subalgebra of A. Hence, $(\widetilde{F}_1, E_1) \cup (\widetilde{F}_2, E_2)$ is not a fuzzy soft UP_s-subalgebra of A. Moreover, $(\widetilde{F}_1, E_1) \vee (\widetilde{F}_2, E_2)$ is not a fuzzy soft UP_s-subalgebra of A.

We can apply this example for check that the OR of two fuzzy soft UP_i-subalgebras (resp., fuzzy soft near UP_s-filters, fuzzy soft UP_s-filters, fuzzy soft UP_i-filters, fuzzy soft UP_s-ideals, fuzzy soft UP_i-ideals) of A are not fuzzy soft UP_i-subalgebra (resp., fuzzy soft near UP_s-filter, fuzzy soft UP_s-filter, fuzzy soft UP_i-filter, fuzzy soft UP_s-ideal, fuzzy soft UP_i-ideal).

We prove that certain distributive laws hold in fuzzy soft set theory with respect to the restricted union, the union, the intersection, and the extended intersection on any fuzzy soft sets.

Theorem 4.5.16 *Let $(\widetilde{F}_i, E_i)$ and $(\widetilde{F}, E)$ be fuzzy soft sets over a common universe U where I is a nonempty set. Then the following properties hold:*

(1) $(\widetilde{F}, E) \Cap (\bigcup_{i\in I}(\widetilde{F}_i, E_i)) = \bigcup_{i\in I}((\widetilde{F}, E) \Cap (\widetilde{F}_i, E_i))$,

(2) $(\bigcup_{i\in I}(\widetilde{F}_i, E_i)) \Cap (\widetilde{F}, E) = \bigcup_{i\in I}((\widetilde{F}_i, E_i) \Cap (\widetilde{F}, E))$,

(3) $(\widetilde{F}, E) \Cup (\bigcap_{i\in I}(\widetilde{F}_i, E_i)) = \bigcap_{i\in I}((\widetilde{F}, E) \Cup (\widetilde{F}_i, E_i))$,

(4) $(\bigcap_{i\in I}(\widetilde{F}_i, E_i)) \Cup (\widetilde{F}, E) = (\widetilde{F}_i, E_i)) \Cup \bigcap_{i\in I}((\widetilde{F}, E)$,

(5) $(\widetilde{F}, E) \cap (\Cup_{i\in I}(\widetilde{F}_i, E_i)) = \Cup_{i\in I}((\widetilde{F}, E) \cap (\widetilde{F}_i, E_i))$,

(6) $(\Cup_{i\in I}(\widetilde{F}_i, E_i)) \cap (\widetilde{F}, E) = \Cup_{i\in I}((\widetilde{F}_i, E_i) \cap (\widetilde{F}, E))$,

(7) $(\widetilde{F}, E) \cup (\Cap_{i\in I}(\widetilde{F}_i, E_i)) = \Cap_{i\in I}((\widetilde{F}, E) \cup (\widetilde{F}_i, E_i))$,

(8) $(\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \cup (\widetilde{\mathrm{F}}, E) = \Cap_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \cup (\widetilde{\mathrm{F}}, E))$,

(9) $(\widetilde{\mathrm{F}}, E) \Cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i))$,

(10) $(\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cap (\widetilde{\mathrm{F}}, E) = \Cup_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \Cap (\widetilde{\mathrm{F}}, E))$,

(11) $(\widetilde{\mathrm{F}}, E) \Cup (\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cap_{i\in I}((\widetilde{\mathrm{F}}, E) \Cup (\widetilde{\mathrm{F}}_i, E_i))$, *and*

(12) $(\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cup (\widetilde{\mathrm{F}}, E) = \Cap_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \Cup (\widetilde{\mathrm{F}}, E))$.

Proof. (1) First, we investigate left hand side of the equality. Suppose that $\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{G}}, E^U)$ is the union of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E^U = \bigcup_{i\in I} E_i$ and for any $e \in E^U$, $\widetilde{\mathrm{G}}[e] = \bigcup_{j\in J} \widetilde{\mathrm{F}}_j[e]$ with $e \in \bigcap_{j\in J} E_j - \bigcup_{k\in I-J} E_k$ where $\emptyset \neq J \subseteq I$. Thus $(\widetilde{\mathrm{F}}, E) \Cap (\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = (\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{G}}, E^U) = (\widetilde{\mathrm{H}}, E^{UI})$. For any $e \in E^{UI} = E \cap E^U \neq \emptyset$, $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e]$ where $E \cap E^U = E \cap (\bigcup_{i\in I} E_i) = \bigcup_{i\in I}(E \cap E_i)$. By considering $\widetilde{\mathrm{G}}$ as piecewise defined function, we have $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cap (\bigcup_{j\in J} \widetilde{\mathrm{F}}_j[e])$ with $e \in \bigcap_{j\in J}(E \cap E_j) - \bigcup_{k\in I-J}(E \cap E_k)$ where $\emptyset \neq J \subseteq I$.

Consider the right hand side of the equality. Suppose that $(\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{I}}_i, E_i^I)$ is the intersection of $(\widetilde{\mathrm{F}}, E)$ and $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E_i^I = E \cap E_i \neq \emptyset$ and for any $e \in E_i^I$, $\widetilde{\mathrm{I}}_i[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_i[e]$. Now, $\bigcup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i)) = \bigcup_{i\in I}(\widetilde{\mathrm{I}}_i, E_i^I) = (\widetilde{\mathrm{J}}, E^{IU})$, where $E^{IU} = \bigcup_{i\in I} E_i^I = \bigcup_{i\in I}(E \cap E_i)$. For any $e \in E^{IU}$, $\widetilde{\mathrm{J}}[e] = \bigcup_{j\in J} \widetilde{\mathrm{I}}_j[e]$ with $e \in \bigcap_{j\in J} E_j^I - \bigcup_{k\in I-J} E_k^I$ where $\emptyset \neq J \subseteq I$. Considering $\widetilde{\mathrm{I}}_i$ as piecewise functions for all $i \in I$, we have $\widetilde{\mathrm{J}}[e] = \bigcup_{j\in J}(\widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_j[e])$ with $e \in \bigcap_{j\in J}(E \cap E_j) - \bigcup_{k\in I-J}(E \cap E_k)$ where $\emptyset \neq J \subseteq I$. By Theorem 3.0.37(1), it is clear that $\widetilde{\mathrm{H}}$ and $\widetilde{\mathrm{J}}$ are same set-valued mapping. Hence, $(\widetilde{\mathrm{F}}, E) \Cap (\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \bigcup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i))$.

(2) By using techniques as in (1) and by Theorem 3.0.37(2), then (2) can is derived.

(3) By using techniques as in (1) and by Theorem 3.0.37(3), then (3) can is derived.

(4) By using techniques as in (1) and by Theorem 3.0.37(4), then (4) can is derived.

(5) First, we investigate left hand side of the equality. Suppose that $\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{G}}, E^{RU})$ is the restricted union of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E^{RU} = \bigcap_{i\in I} E_i \neq \emptyset$ and for any $e \in E^{RU}$, $\widetilde{\mathrm{G}}[e] = \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e]$. Thus $(\widetilde{\mathrm{F}}, E) \cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = (\widetilde{\mathrm{F}}, E)\cap(\widetilde{\mathrm{G}}, E^{RU}) = (\widetilde{\mathrm{H}}, E^{RUEI})$. For any $e \in E^{RUEI} = E\cup E^{RU}$, we have

$$\widetilde{\mathrm{H}}[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E \setminus E^{RU} \\ \widetilde{\mathrm{G}}[e] & \text{if } e \in E^{RU} \setminus E \\ \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e] & \text{if } e \in E \cap E^{RU}. \end{cases}$$

By taking into account the definition of $\widetilde{\mathrm{G}}$ along with $\widetilde{\mathrm{H}}$, we can write

$$\widetilde{\mathrm{H}}[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E \setminus (\bigcap_{i\in I} E_i) \\ \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e] & \text{if } e \in (\bigcap_{i\in I} E_i) \setminus E \\ \widetilde{\mathrm{F}}[e] \cap (\bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e]) & \text{if } e \in E \cap (\bigcap_{i\in I} E_i). \end{cases}$$

Consider the right hand side of the equality. Suppose that $(\widetilde{\mathrm{F}}, E) \cap (\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{I}}_i, E_i^{EI})$ is the extended intersection of $(\widetilde{\mathrm{F}}, E)$ and $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then for any $e \in E_i^{EI} = E \cup E_i$, we have

$$\widetilde{\mathrm{I}}_i[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E \setminus E_i \\ \widetilde{\mathrm{F}}_i[e] & \text{if } e \in E_i \setminus E \\ \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_i[e] & \text{if } e \in E \cap E_i. \end{cases}$$

Now, $\Cup_{i\in I}((\widetilde{\mathrm{F}}, E)\cap(\widetilde{\mathrm{F}}_i, E_i))$=$\Cup_{i\in I}(\widetilde{\mathrm{I}}_i, E_i^{EI}) = (\widetilde{\mathrm{J}}, E^{EIRU})$ where $E^{EIRU} = \bigcap_{i\in I} E_i^I$ $= \bigcap_{i\in I}(E \cup E_i) = E \cup (\bigcap_{i\in I} E_i) \neq \emptyset$. For any $e \in E^{EIRU}$, $\widetilde{\mathrm{J}}[e] = \bigcup_{i\in I} \widetilde{\mathrm{I}}_i[e]$. By taking into account the properties of operations in set theory and considering $\widetilde{\mathrm{I}}_i$

as piecewise defined functions for all $i \in I$, we have

$$\widetilde{\mathrm{J}}[e] = \begin{cases} \bigcup_{i\in I} \widetilde{\mathrm{F}}[e] & \text{if } e \in E \setminus (\bigcap_{i\in I} E_i) \\ \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e] & \text{if } e \in (\bigcap_{i\in I} E_i) \setminus E \\ \bigcup_{i\in I}(\widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_i[e]) & \text{if } e \in E \cap (\bigcap_{i\in I} E_i). \end{cases}$$

And so

$$\widetilde{\mathrm{J}}[e] = \begin{cases} \widetilde{\mathrm{F}}[e] & \text{if } e \in E \setminus (\bigcap_{i\in I} E_i) \\ \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e] & \text{if } e \in (\bigcap_{i\in I} E_i) \setminus E \\ \bigcup_{i\in I}(\widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_i[e]) & \text{if } e \in E \cap (\bigcap_{i\in I} E_i). \end{cases}$$

By Theorem 3.0.37(1), it is clear that $\widetilde{\mathrm{H}}$ and $\widetilde{\mathrm{J}}$ are same set-valued mapping. Hence, $(\widetilde{\mathrm{F}}, E) \cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}((\widetilde{\mathrm{F}}, E) \cap (\widetilde{\mathrm{F}}_i, E_i))$.

(6) By using techniques as in (5) and by Theorem 3.0.37(2), then (6) can is derived.

(7) By using techniques as in (5) and by Theorem 3.0.37(3), then (7) can is derived.

(8) By using techniques as in (5) and by Theorem 3.0.37(4), then (8) can is derived.

(9) First, we investigate left hand side of the equality. Suppose that $\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{G}}, E^{RU})$ is the restricted union of $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E^{RU} = \bigcap_{i\in I} E_i \neq \emptyset$ and for any $e \in E^{RU}$, $\widetilde{\mathrm{G}}[e] = \bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e]$. Thus $(\widetilde{\mathrm{F}}, E) \Cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = (\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{G}}, E^{RU}) = (\widetilde{\mathrm{H}}, E^{RUI})$. For any $e \in E^{RUI} = E \cap E^{RU} = E \cap (\bigcap_{i\in I} E_i) \neq \emptyset$, we have $\widetilde{\mathrm{H}}[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{G}}[e] = \widetilde{\mathrm{F}}[e] \cap (\bigcup_{i\in I} \widetilde{\mathrm{F}}_i[e])$.

Consider the right hand side of the equality. Suppose that $(\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i) = (\widetilde{\mathrm{I}}_i, E_i^I)$ is the intersection of $(\widetilde{\mathrm{F}}, E)$ and $(\widetilde{\mathrm{F}}_i, E_i)$ for all $i \in I$. Then $E_i^I = E \cap E_i \neq \emptyset$ and for any $e \in E_i^I$, $\widetilde{\mathrm{I}}_i[e] = \widetilde{\mathrm{F}}[e] \cap \widetilde{\mathrm{F}}_i[e]$. Now, $\Cup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}(\widetilde{\mathrm{I}}_i, E_i^I) = (\widetilde{\mathrm{J}}, E^{IRU})$, where $E^{IRU} = \bigcap_{i\in I} E_i^I = \bigcap_{i\in I}(E \cap E_i) \neq \emptyset$. For any $e \in$

E^{IRU}, $\widetilde{\mathrm{J}}[e] = \bigcup_{j\in J}\widetilde{\mathrm{I}}_j[e] = \bigcup_{j\in J}(\widetilde{\mathrm{F}}[e]\cap\widetilde{\mathrm{F}}_i[e])$. Since $\bigcap_{i\in I}(E\cap E_i) = E\cap(\bigcap_{i\in I} E_i)$, we have $E^{IRU} = E^{RUI}$. By Theorem 3.0.37(1), it is clear that $\widetilde{\mathrm{H}}$ and $\widetilde{\mathrm{J}}$ are same set-valued mapping. Hence, $(\widetilde{\mathrm{F}}, E)\Cap(\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}((\widetilde{\mathrm{F}}, E)\Cap(\widetilde{\mathrm{F}}_i, E_i))$.

(10) By using techniques as in (9) and by Theorem 3.0.37(2), then (10) can is derived.

(11) By using techniques as in (9) and by Theorem 3.0.37(3), then (11) can is derived.

(12) By using techniques as in (9) and by Theorem 3.0.37(4), then (12) can is derived. □

CHAPTER V

CONCLUSIONS

From the study, we get the following results.

1. Every UP_i-subalgebra of A is a UP_s-subalgebra of A.
2. Every near UP_s-filter of A is a UP_s-subalgebra of A.
3. Every near UP_i-filter of A is a UP_i-subalgebra of A.
4. Every near UP_i-filter of A is a near UP_s-filter of A.
5. Every UP_s-filter of A is a near UP_s-filter of A.
6. Every UP_i-filter of A is a near UP_i-filter of A.
7. Every UP_i-filter of A is a UP_s-filter of A.
8. Every UP_s-ideal of A is a UP_s-filter of A.
9. Every UP_i-ideal of A is a UP_i-filter of A.
10. Every UP_i-ideal of A is a UP_s-ideal of A.
11. Every strongly UP_s-ideal of A is a UP_s-ideal of A.
12. Every strongly UP_i-ideal of A is a UP_i-ideal of A.
13. Strongly UP_s-ideals and strongly UP_i-ideals coincide in A and it is only A.
14. The intersection of any nonempty family of fuzzy UP_s-subalgebras of A is also a fuzzy UP_s-subalgebra of A.
15. A nonempty subset S of A is a UP_s-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_s-subalgebra of A.

16. The intersection of any nonempty family of fuzzy $\mathrm{UP_i}$-subalgebras of A is also a fuzzy $\mathrm{UP_i}$-subalgebra of A.

17. A nonempty subset S of A is a $\mathrm{UP_i}$-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_i}$-subalgebra of A.

18. The intersection of any nonempty family of fuzzy near $\mathrm{UP_s}$-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near $\mathrm{UP_s}$-filter.

19. A nonempty subset S of A is a near $\mathrm{UP_s}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy near $\mathrm{UP_s}$-filter of A.

20. The intersection of any nonempty family of fuzzy near $\mathrm{UP_i}$-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near $\mathrm{UP_i}$-filter.

21. The union of any nonempty family of fuzzy near $\mathrm{UP_i}$-filters of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is also a fuzzy near $\mathrm{UP_i}$-filter.

22. A nonempty subset S of A is a near $\mathrm{UP_i}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy near $\mathrm{UP_i}$-filter of A.

23. Every fuzzy near $\mathrm{UP_s}$-filter of an f-UP-semigroup is a fuzzy $\mathrm{UP_s}$-subalgebra.

24. Every fuzzy near $\mathrm{UP_i}$-filter of an f-UP-semigroup is a fuzzy $\mathrm{UP_i}$-subalgebra.

25. The intersection of any nonempty family of fuzzy $\mathrm{UP_s}$-filters of A is also a fuzzy $\mathrm{UP_s}$-filter of A.

26. A nonempty subset S of A is a $\mathrm{UP_s}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_s}$-filter of A.

27. The intersection of any nonempty family of fuzzy $\mathrm{UP_i}$-filters of A is also a fuzzy $\mathrm{UP_i}$-filter of A.

28. A nonempty subset S of A is a $\mathrm{UP_i}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_i}$-filter of A.

29. Every fuzzy $\mathrm{UP_s}$-filter of an f-UP-semigroup is a fuzzy near $\mathrm{UP_s}$-filter.

30. Every fuzzy $\mathrm{UP_i}$-filter of an f-UP-semigroup is a fuzzy near $\mathrm{UP_i}$-filter.

31. The intersection of any nonempty family of fuzzy $\mathrm{UP_s}$-ideals of A is also a fuzzy $\mathrm{UP_s}$-ideal of A.

32. A nonempty subset S of A is a $\mathrm{UP_s}$-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_s}$-ideal of A.

33. The intersection of any nonempty family of fuzzy $\mathrm{UP_i}$-ideals of A is also a fuzzy $\mathrm{UP_i}$-ideal of A.

34. A nonempty subset S of A is a $\mathrm{UP_i}$-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy $\mathrm{UP_i}$-ideal of A.

35. Every fuzzy $\mathrm{UP_s}$-ideal of A is a fuzzy $\mathrm{UP_s}$-filter of A.

36. Every fuzzy $\mathrm{UP_i}$-ideal of A is a fuzzy $\mathrm{UP_i}$-filter of A.

37. Fuzzy strongly $\mathrm{UP_s}$-ideals, fuzzy strongly $\mathrm{UP_i}$-ideals, and constant fuzzy sets coincide in A.

38. The intersection and union of any nonempty family of fuzzy strongly $\mathrm{UP_s}$-ideals of A are also a fuzzy strongly $\mathrm{UP_s}$-ideal of A.

39. The intersection and union of any nonempty family of fuzzy strongly $\mathrm{UP_i}$-ideals of A are also a fuzzy strongly $\mathrm{UP_i}$-ideal of A.

40. A nonempty subset S of A is a strongly $\mathrm{UP_s}$-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly $\mathrm{UP_s}$-ideal of A.

41. A nonempty subset S of A is a strongly $\mathrm{UP_i}$-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy strongly $\mathrm{UP_i}$-ideal of A.

42. Every fuzzy strongly $\mathrm{UP_s}$-ideal (fuzzy strongly $\mathrm{UP_i}$-ideal) of A is a fuzzy $\mathrm{UP_s}$-ideal and a fuzzy $\mathrm{UP_i}$-ideal of A.

43. If F is a fuzzy set in A satisfying the condition (4.3.3), then F satisfies the condition (4.3.1).

44. If F is a fuzzy set in A satisfying the condition (4.3.2), then F satisfies the condition (4.3.4).

45. If F is a fuzzy UP-subalgebra of A satisfying the condition

$$(\forall x, y \in A)(x \cdot y \neq 0 \Rightarrow \mathrm{f_F}(x) \geq \mathrm{f_F}(y)), \tag{4.3.5}$$

then F is a fuzzy near UP-filter of A.

46. If F is a fuzzy set in A satisfying the condition (4.3.6), then F satisfies the condition (4.3.2).

47. If F is a fuzzy near UP-filter of A satisfying the condition

$$(\forall x, y \in A)(\mathrm{f_F}(x \cdot y) = \mathrm{f_F}(y)), \tag{4.3.7}$$

then F is a fuzzy UP-filter of A.

48. Let A be a UP-algebra satisfying the condition

$$(\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x)). \tag{4.3.10}$$

If F is a fuzzy set in A satisfying the condition (4.3.9), then F satisfies the condition (4.3.8).

49. If F is a fuzzy set in A satisfying the condition (4.3.9), then F satisfies the condition (4.3.6).

50. If F is a fuzzy UP-filter of A satisfying the condition

$$(\forall x, y, z \in A)(\mathrm{f_F}(y \cdot (x \cdot z)) = \mathrm{f_F}(x \cdot (y \cdot z))), \tag{4.3.11}$$

then F is a fuzzy UP-ideal of A.

51. If F is a fuzzy set in A satisfying the condition

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{f_F}(z) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}), \tag{4.3.13}$$

then F satisfies the condition (4.3.3).

52. If F is a fuzzy set in A satisfying the condition

$$(\forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{f_F}(z) \geq \mathrm{f_F}(y)), \tag{4.3.14}$$

then F satisfies the condition (4.3.3).

53. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.3) and (3.0.14), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

54. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A.

55. The extended intersection of two fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra. Moreover, the intersection of two fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra.

56. The union of two fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra if sets of statistics of two fuzzy soft $\mathrm{UP_s}$-subalgebras are disjoint.

57. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$

in A satisfies the conditions (4.3.3) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-subalgebra of A.

58. Every e-fuzzy soft $\mathrm{UP_i}$-subalgebra of A is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra. Moreover, every fuzzy soft $\mathrm{UP_i}$-subalgebra of A is a fuzzy soft $\mathrm{UP_s}$-subalgebra.

59. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-subalgebra of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_i}$-subalgebra of A.

60. The extended intersection of two fuzzy soft $\mathrm{UP_i}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_i}$-subalgebra. Moreover, the intersection of two fuzzy soft $\mathrm{UP_i}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_i}$-subalgebra.

61. The union of two fuzzy soft $\mathrm{UP_i}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_i}$-subalgebra if sets of statistics of two fuzzy soft $\mathrm{UP_i}$-subalgebras are disjoint.

62. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.2) and (3.0.14), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A.

63. Every e-fuzzy soft near $\mathrm{UP_s}$-filter of A is an e-fuzzy soft $\mathrm{UP_s}$-subalgebra. Moreover, every fuzzy soft near $\mathrm{UP_s}$-filter of A is a fuzzy soft $\mathrm{UP_s}$-subalgebra.

64. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-subalgebra of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.5), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A.

65. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A.

66. The extended intersection of two fuzzy soft near $\mathrm{UP_s}$-filters of A is also a fuzzy soft near $\mathrm{UP_s}$-filter. Moreover, the intersection of two fuzzy soft near $\mathrm{UP_s}$-filters of A is also a fuzzy soft near $\mathrm{UP_s}$-filter.

67. The union of two fuzzy soft near $\mathrm{UP_s}$-filters of A is also a fuzzy soft near $\mathrm{UP_s}$-filter if sets of statistics of two fuzzy soft near $\mathrm{UP_s}$-filters are disjoint.

68. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.2) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A.

69. Every e-fuzzy soft near $\mathrm{UP_i}$-filter of A is an e-fuzzy soft near $\mathrm{UP_s}$-filter. Moreover, every fuzzy soft near $\mathrm{UP_i}$-filter of A is a fuzzy soft near $\mathrm{UP_s}$-filter.

70. Every e-fuzzy soft near $\mathrm{UP_i}$-filter of A is an e-fuzzy soft $\mathrm{UP_i}$-subalgebra. Moreover, every fuzzy soft near $\mathrm{UP_i}$-filter of A is a fuzzy soft $\mathrm{UP_i}$-subalgebra.

71. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-subalgebra of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.5), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A.

72. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A.

73. The extended intersection of two fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter. Moreover, the intersection of two fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter.

74. The union of two fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter. Moreover, the restricted union of two fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter.

75. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.6) and (3.0.14), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-filter of A.

76. Every e-fuzzy soft $\mathrm{UP_s}$-filter of A is an e-fuzzy soft near $\mathrm{UP_s}$-filter. Moreover, every fuzzy soft $\mathrm{UP_s}$-filter of A is a fuzzy soft near $\mathrm{UP_s}$-filter.

77. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_s}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.7), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-filter of A.

78. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_s}$-filter of A.

79. The extended intersection of two fuzzy soft $\mathrm{UP_s}$-filters of A is also a fuzzy soft $\mathrm{UP_s}$-filter. Moreover, the intersection of two fuzzy soft $\mathrm{UP_s}$-filters of A is also a fuzzy soft $\mathrm{UP_s}$-filter.

80. The union of two fuzzy soft $\mathrm{UP_s}$-filters of A is also a fuzzy soft $\mathrm{UP_s}$-filter if sets of statistics of two fuzzy soft $\mathrm{UP_s}$-filters are disjoint.

81. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.6) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-filter of A.

82. Every e-fuzzy soft $\mathrm{UP_i}$-filter of A is an e-fuzzy soft $\mathrm{UP_s}$-filter. Moreover, every fuzzy soft $\mathrm{UP_i}$-filter of A is a fuzzy soft $\mathrm{UP_s}$-filter.

83. Every e-fuzzy soft $\mathrm{UP_i}$-filter of A is an e-fuzzy soft near $\mathrm{UP_i}$-filter. Moreover, every fuzzy soft $\mathrm{UP_i}$-filter of A is a fuzzy soft near $\mathrm{UP_i}$-filter.

84. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft near $\mathrm{UP_i}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.7), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-filter of A.

85. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-filter of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_i}$-filter of A.

86. The extended intersection of two fuzzy soft $\mathrm{UP_i}$-filters of A is also a fuzzy soft $\mathrm{UP_i}$-filter. Moreover, the intersection of two fuzzy soft $\mathrm{UP_i}$-filters of A is also a fuzzy soft $\mathrm{UP_i}$-filter.

87. The union of two fuzzy soft $\mathrm{UP_i}$-filters of A is also a fuzzy soft $\mathrm{UP_i}$-filter if sets of statistics of two fuzzy soft $\mathrm{UP_i}$-filters are disjoint.

88. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.8) and (3.0.14), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-ideal of A.

89. Every e-fuzzy soft $\mathrm{UP_s}$-ideal of A is an e-fuzzy soft $\mathrm{UP_s}$-filter. Moreover, every fuzzy soft $\mathrm{UP_s}$-ideal of A is a fuzzy soft $\mathrm{UP_s}$-filter.

90. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.11), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-ideal of A.

91. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_s}$-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_s}$-ideal of A.

92. The extended intersection of two fuzzy soft $\mathrm{UP_s}$-ideals of A is also a fuzzy soft $\mathrm{UP_s}$-ideal. Moreover, the intersection of two fuzzy soft $\mathrm{UP_s}$-ideals of A is also a fuzzy soft $\mathrm{UP_s}$-ideal.

93. The union of two fuzzy soft $\mathrm{UP_s}$-ideals of A is also a fuzzy soft $\mathrm{UP_s}$-ideal if sets of statistics of two fuzzy soft $\mathrm{UP_s}$-ideals are disjoint.

94. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.8) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-ideal of A.

95. Every e-fuzzy soft $\mathrm{UP_i}$-ideal of A is an e-fuzzy soft $\mathrm{UP_s}$-ideal. Moreover, every fuzzy soft $\mathrm{UP_i}$-ideal of A is a fuzzy soft $\mathrm{UP_s}$-ideal.

96. Every e-fuzzy soft $\mathrm{UP_i}$-ideal of A is an e-fuzzy soft $\mathrm{UP_i}$-filter. Moreover, every fuzzy soft $\mathrm{UP_i}$-ideal of A is a fuzzy soft $\mathrm{UP_i}$-filter.

97. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-filter of A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the condition (4.3.11), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-ideal of A.

98. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft $\mathrm{UP_i}$-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft $\mathrm{UP_i}$-ideal of A.

99. The extended intersection of two fuzzy soft $\mathrm{UP_i}$-ideals of A is also a fuzzy soft $\mathrm{UP_i}$-ideal. Moreover, the intersection of two fuzzy soft $\mathrm{UP_i}$-ideals of A is also a fuzzy soft $\mathrm{UP_i}$-ideal.

100. The union of two fuzzy soft $\mathrm{UP_i}$-ideals of A is also a fuzzy soft $\mathrm{UP_i}$-ideal if sets of statistics of two fuzzy soft $\mathrm{UP_i}$-ideals are disjoint.

101. Every e-fuzzy soft strongly $\mathrm{UP_s}$-ideal of A is an e-fuzzy soft $\mathrm{UP_s}$-ideal. Moreover, every fuzzy soft strongly $\mathrm{UP_s}$-ideal of A is a fuzzy soft $\mathrm{UP_s}$-ideal.

102. e-fuzzy soft strongly $\mathrm{UP_s}$-ideals and e-constant fuzzy soft sets coincide in A. Moreover, fuzzy soft strongly $\mathrm{UP_s}$-ideals and constant fuzzy soft sets coincide in A.

103. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.12) (or (4.3.13) or (4.3.14)) and (3.0.14), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly $\mathrm{UP_s}$-ideal of A.

104. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly $\mathrm{UP_s}$-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft strongly $\mathrm{UP_s}$-ideal of A.

105. The extended intersection of two fuzzy soft strongly $\mathrm{UP_s}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal. Moreover, the intersection of two fuzzy soft strongly $\mathrm{UP_s}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal.

106. The union of two fuzzy soft strongly $\mathrm{UP_s}$-ideals is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal. Moreover, the restricted union of two fuzzy soft strongly $\mathrm{UP_s}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal.

107. Every e-fuzzy soft strongly $\mathrm{UP_i}$-ideal of A is an e-fuzzy soft $\mathrm{UP_i}$-ideal. Moreover, every fuzzy soft strongly $\mathrm{UP_i}$-ideal of A is a fuzzy soft $\mathrm{UP_i}$-ideal.

108. e-fuzzy soft strongly $\mathrm{UP_i}$-ideals and e-constant fuzzy soft sets coincide in A. Moreover, fuzzy soft strongly $\mathrm{UP_i}$-ideals and constant fuzzy soft sets coincide in A.

109. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft set over A such that for all $e \in E$, a fuzzy set $\widetilde{\mathrm{F}}[e]$ in A satisfies the conditions (4.3.12) (or (4.3.13) or (4.3.14)) and (3.0.15), then $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly $\mathrm{UP_i}$-ideal of A.

110. If $(\widetilde{\mathrm{F}}, E)$ is a fuzzy soft strongly $\mathrm{UP_i}$-ideal of A and $\emptyset \neq E^* \subseteq E$, then $(\widetilde{\mathrm{F}}|_{E^*}, E^*)$ is a fuzzy soft strongly $\mathrm{UP_i}$-ideal of A.

111. The extended intersection of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal. Moreover, the intersection of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal.

112. The union of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_s}$-ideal. Moreover, the restricted union of two fuzzy soft strongly $\mathrm{UP_i}$-ideals of A is also a fuzzy soft strongly $\mathrm{UP_i}$-ideal.

113. The restricted union of family of fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter.

114. The union of family of fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter.

115. The intersection of family of fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra.

116. The extended intersection of family of fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra.

117. The AND of family of fuzzy soft $\mathrm{UP_s}$-subalgebras of A is also a fuzzy soft $\mathrm{UP_s}$-subalgebra.

118. The OR of family of fuzzy soft near $\mathrm{UP_i}$-filters of A is also a fuzzy soft near $\mathrm{UP_i}$-filter.

119. Let $(\widetilde{\mathrm{F}}_i, E_i)$ and $(\widetilde{\mathrm{F}}, E)$ be fuzzy soft sets over a common universe U where I is a nonempty set. Then the following properties hold:

(1) $(\widetilde{\mathrm{F}}, E) \Cap (\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \bigcup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i))$,

(2) $(\bigcup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cap (\widetilde{\mathrm{F}}, E) = \bigcup_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \Cap (\widetilde{\mathrm{F}}, E))$,

(3) $(\widetilde{\mathrm{F}}, E) \Cup (\bigcap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \bigcap_{i\in I}((\widetilde{\mathrm{F}}, E) \Cup (\widetilde{\mathrm{F}}_i, E_i))$,

(4) $(\bigcap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cup (\widetilde{\mathrm{F}}, E) = (\widetilde{\mathrm{F}}_i, E_i)) \Cup \bigcap_{i\in I}((\widetilde{\mathrm{F}}, E)$,

(5) $(\widetilde{\mathrm{F}}, E) \cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}((\widetilde{\mathrm{F}}, E) \cap (\widetilde{\mathrm{F}}_i, E_i))$,

(6) $(\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \cap (\widetilde{\mathrm{F}}, E) = \Cup_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \cap (\widetilde{\mathrm{F}}, E))$,

(7) $(\widetilde{\mathrm{F}}, E) \cup (\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cap_{i\in I}((\widetilde{\mathrm{F}}, E) \cup (\widetilde{\mathrm{F}}_i, E_i))$,

(8) $(\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \cup (\widetilde{\mathrm{F}}, E) = \Cap_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \cup (\widetilde{\mathrm{F}}, E))$,

(9) $(\widetilde{\mathrm{F}}, E) \Cap (\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cup_{i\in I}((\widetilde{\mathrm{F}}, E) \Cap (\widetilde{\mathrm{F}}_i, E_i))$,

(10) $(\Cup_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cap (\widetilde{\mathrm{F}}, E) = \Cup_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \Cap (\widetilde{\mathrm{F}}, E))$,

(11) $(\widetilde{\mathrm{F}}, E) \Cup (\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) = \Cap_{i\in I}((\widetilde{\mathrm{F}}, E) \Cup (\widetilde{\mathrm{F}}_i, E_i))$, and

(12) $(\Cap_{i\in I}(\widetilde{\mathrm{F}}_i, E_i)) \Cup (\widetilde{\mathrm{F}}, E) = \Cap_{i\in I}((\widetilde{\mathrm{F}}_i, E_i) \Cup (\widetilde{\mathrm{F}}, E))$.

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