HESITANT FUZZY SOFT SETS OVER UP-ALGEBRAS

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บทคัดย่อ

ในงานวิจัยนี้ เราขยายผลจากแนวคิดของเซตวิภัชนัยฮีซิแตนท์บนพีชคณิตยูพีไปยังเซตอ่อนวิภัชนัยฮีซิแตนท์เหนือพีชคณิตยูพี โดยการรวมแนวคิดระหว่างเซตวิภัชนัยฮีซิแตนท์กับเซตอ่อน เราได้แนวคิดของ(ปฏิ)ไอดีลยูพีอย่างเข้มอ่อนวิภัชนัยฮีซิแตนท์ (ปฏิ)ไอดีลยูพีอ่อนวิภัชนัยฮีซิแตนท์ (ปฏิ)ตัวกรองยูพีอ่อนวิภัชนัยฮีซิแตนท์ และ(ปฏิ)พีชคณิตย่อยยูพีอ่อนวิภัชนัยฮีซิแตนท์เหนือพีชคณิตยูพี และเราจัดหาสมบัติบางอย่างของแนวคิดข้างต้น นอกจากนี้ เราพิจารณาการดำเนินการบางอย่างบนเซตอ่อนวิภัชนัยฮีซิแตนท์และตรวจสอบผลลัพธ์ของการดำเนินการ

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ABSTRACT

In this research, we extend the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the concepts of hesitant fuzzy sets and soft sets. We obtain the concepts of (anti-)hesitant fuzzy soft strongly UP-ideals, (anti-)hesitant fuzzy soft UP-ideals, (anti-)hesitant fuzzy soft UP-filters and (anti-)hesitant fuzzy soft UP-subalgebras over UP-algebras, and provide some properties of them. Furthermore, we discuss some operations on hesitant fuzzy soft sets and we examine some results of them.

LIST OF CONTENTS

LIST OF FIGURES

Figures **Page**

CHAPTER I

INTRODUCTION

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [12], BCI-algebras [13], KU-algebras [30], UP-algebras [10], and so forth. Two classes of logical algebras, BCK and BCI-algebras were introduced by Imai and Iséki [12, 13] in 1966 and they have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The branch of the logical algebra, a UP-algebra was introduced by Iampan [10] in 2017, and it is known that the class of KU-algebras [30] is a proper subclass of the class of UP-algebras. It has been examined by several researchers, for example, Somjanta et al. [36] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was studied by Kesorn et al. [17], the concept of Q-fuzzy sets in UP-algebras was investigated by Tanamoon et al. [39], Kaijae et al. [16] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, Sripaeng et al. [38] introduced the notion anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the concept of fuzzy translations of a fuzzy set was applied to UP-algebras by Guntasow et al. [9], the notion of $\mathcal{N}$-fuzzy sets in UP-algebras was studied by Songsaeng and Iampan [37], Senapati et al. [34, 35] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [31] introduced the notion of proper UP-filters in UP-algebras, etc.

There are many sophisticated problems, such as economics, engineering, environment, social science, medical science, etc., that we cannot successfully use classical methods because of various uncertainties typical for those problems. There are several theories: theory of probability, theory of fuzzy sets [43], theory of intuitionistic fuzzy sets [3, 4], theory of vague sets [7], theory of interval mathematics, [4, 8], and theory of rough sets [28] which can be considered as mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties as pointed out in [23]. The reason for these difficulties possibly is the inadequacy of the parametrization tool of the theories. Molodtsov [23] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Maji et al. [22] described the application of soft set theory to a decision making problem. Maji et al. [20] and Ali et al. [1] studied several operations on the theory of soft sets. Chen et al. [6] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The concept of hesitant fuzzy soft sets, as a generalization of the standard soft sets, was introduced by Babitha and John [5] and Wang et al. [41], and an application of hesitant fuzzy soft sets in a decision making problem is presented.

The main purpose of this research is to extend the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the concept of hesitant fuzzy sets and soft sets. The main results obtain the concepts of (anti-)hesitant fuzzy soft strongly UP-ideals, (anti-)hesitant fuzzy soft UP-ideals, (anti-)hesitant fuzzy soft UP-filters, and (anti-)hesitant fuzzy soft UP-subalgebras of UP-algebras, and also obtain some properties of them. Moreover, we apply some operations to our main results.

CHAPTER II

REVIEW OF RELATED LITERATURE AND RESEARCH

A fuzzy set f of a set X is a function from X to a closed interval $[0, 1]$. The concept of a fuzzy set of a set was first considered by Zadeh [43] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. The hesitant fuzzy set on a reference set X is defined in term of a function that when applied to X return a subset of $[0, 1]$. Torra [40] is the first researcher who, introduced one of the generalized fuzzy sets, called a hesitant fuzzy set on a set in 2010.

A soft set over a universe set is a parametized family of subsets of the universe set. Molodtsov [23] introduced the concept of soft sets over a universe set in 1999. The soft set theory was a new mathematical tool to handle uncertainty but the classical soft sets are not appropriate to deal with some imprecisions. The soft set model consequently has been combined with other mathematical models. For example, fuzzy soft sets [18] are based on a combination of fuzzy sets and soft sets, intuitionistic fuzzy soft sets [19, 21] are combined by intuitionistic fuzzy sets and soft sets, interval-valued fuzzy soft sets [42] are combined by interval-valued fuzzy sets and soft sets, and so on. The concept of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets was studied by Babitha and John [5] and Wang et al. [41]. They also presented an application of hesitant fuzzy soft sets in a decision making problem. Moreover, basic operations such as intersection, union, AND, OR, compliment were defined, and De Morgans law was also proved. There are some researchers, such as Jun et al. [14], Alshehri and Alshehri [2], applied hesitant fuzzy soft set theory to BCK/BCI-algebras.

Iampan [10] is the first one, who introduced the branch of algebraic structure, which is called a UP-algebra in 2017, and proved that the notion of UP-algebras is a generalization of KU-algebras. Iampan also introduced the concepts of UP-subalgebras of UP-algebras, UP-ideals of UP-algebras, congruences on UP-algebras, and UP-homomorphisms in UP-algebras, and investigated some related properties of them. Somjanta et al. [36] introduced the concepts of UP-filters of UP-algebras, and applied fuzzy set theory to UP-algebras. The notions of fuzzy UP-subalgebras of UP-algebras, fuzzy UP-ideals of UP-algebras, and fuzzy UP-filters of UP-algebras were studied by them. Moreover, they introduced the notions of upper t-(strong) level subsets and lower t-(strong) level subsets from some fuzzy sets, and gave its characterizations. Poungsumpao et al. [29] investigated the notions of fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras in term of upper t-(strong) level subsets and lower t-(strong) level subsets of a fuzzy set. They also proved that every fuzzy UP-ideal of UP-algebras is a fuzzy UP-subalgebra of UP-algebras. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras were studied by Kaijae et al. [16]. They proved that every anti-fuzzy UP-ideal of UP-algebras is an anti-fuzzy UP-subalgebra of UP-algebras, and also discussed the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Furthermore, they introduced the notions of Cartesian product and dot product of fuzzy sets, and applied anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras in the Cartesian product of UP-algebras. Guntasow et al. [9] introduced the concept of strongly UP-ideals of UP-algebras, and they proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. By the way, they studied the notion of fuzzy strongly UP-ideals of UP-algebras, and they also proved the generalization that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

The first researchers who, extended the concept of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras are Mosrijai et al. [25]. They investigated the notions of hesitant fuzzy UP-subalgebras of UP-algebras, hesitant fuzzy UP-filters of UP-algebras, hesitant fuzzy UP-ideals of UP-algebras, and hesitant fuzzy strongly UP-ideals of UP-algebras. Satirad et al. [33] considered level subsets of a hesitant fuzzy set on UP-algebras. The concepts of partial constant hesitant fuzzy sets and new types of hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [27, 26]. To the best of our knowledge, we are the first one that extend the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the concepts of hesitant fuzzy sets and soft sets.

CHAPTER III

PRELIMINARIES

3.1 Fundamentals

In the chapter, we give the definition of a UP-algebra that was introduced by Iampan [10].

Definition 3.1.1 An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

Iampan also proved that the notion of UP-algebras is a generalization of KU-algebras. Examples of a UP-algebra have been examined by several researchers, for example, the power UP-algebra of type 1 [10], the power UP-algebra of type 2 [10], the generalized power UP-algebra of type 1 with respect to Ω [32], the generalized power UP-algebra of type 2 with respect to Ω [32], and so forth.

Example 3.1.2 Let X be a universal set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A' = A' \cap B = B - A$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*.

Example 3.1.3 Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 3.1.4 Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω.

Example 3.1.5 Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω.

In particular, $(\mathcal{P}_\Omega(X), \cdot, \emptyset)$ is the power UP-algebra of type 1, and $(\mathcal{P}^\Omega(X), *, X)$ is the power UP-algebra of type 2.

Example 3.1.6 [24] Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	3
2	0	0	0	3	3
3	0	0	0	0	3
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra which is not a KU-algebra because $(0 \cdot 2)((2 \cdot 4) \cdot (0 \cdot 4)) = 2 \cdot (3 \cdot 4) = 2 \cdot 3 = 3 \neq 0$ (see the definition in [30]).

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 3.1.7 *[10, 11] In a UP-algebra A, the following properties hold:*

(1) $(\forall x \in A)(x \cdot x = 0)$,

(2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, *and*

(13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

On a UP-algebra $A = (A, \cdot, 0)$, Iampan [10] defined a binary relation $\leq$ on A as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

The following definition is the definition of special subsets in UP-algebras.

Definition 3.1.8 [10, 36, 9] A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$.

(2) a *UP-filter* of A if

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.

(3) a *UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.

(4) a *strongly UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [9] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

CHAPTER IV

HESITANT FUZZY SETS

4.1 Hesitant fuzzy sets

The concept of a hesitant fuzzy set on a set was first considered by Torra [40] in 2010.

Definition 4.1.1 Let X be a reference set. A *hesitant fuzzy set* on X is defined in term of a function $\mathrm{h_H}$ that when applied to X return a subset of $[0,1]$, that is, $\mathrm{h_H}\colon X \to \mathcal{P}([0,1])$. A hesitant fuzzy set $\mathrm{h_H}$ can also be viewed as the following mathematical representation:

$$\mathrm{H} := \{(x, \mathrm{h_H}(x)) \mid x \in X\},$$

where $\mathrm{h_H}(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the elements $x \in X$ to the set H. We say that a hesitant fuzzy set H on X is a constant hesitant fuzzy set if its function $\mathrm{h_H}$ is constant.

Mosrijai et al. [25] introduced the concept the complement of hesitant fuzzy set on UP-algebras and also introduced the concept of hesitant fuzzy UP-subalgebras, hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals, and hesitant fuzzy strongly UP-ideals of UP-algebras in 2017.

Definition 4.1.2 Let H be a hesitant fuzzy set on A. The hesitant fuzzy set $\overline{\mathrm{H}}$ defined by $(\forall x \in A)(\mathrm{h}_{\overline{\mathrm{H}}}(x) = [0,1] - \mathrm{h_H}(x))$, is said to be the *complement* of H on A.

Remark 4.1.3 For all hesitant fuzzy set H on A, we have $\mathrm{H} = \overline{\overline{\mathrm{H}}}$.

Definition 4.1.4 [25] A hesitant fuzzy set H on A is called

(1) a *hesitant fuzzy UP-subalgebra* of A if it satisfies the following property: $(\forall x, y \in A)(\mathrm{h}_{\mathrm{H}}(x \cdot y) \supseteq \mathrm{h}_{\mathrm{H}}(x) \cap \mathrm{h}_{\mathrm{H}}(y))$.

(2) a *hesitant fuzzy UP-filter* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x))$, and

(2) $(\forall x, y \in A)(\mathrm{h}_{\mathrm{H}}(y) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x))$.

(3) a *hesitant fuzzy UP-ideal* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x))$, and

(2) $(\forall x, y, z \in A)(\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y))$.

(4) a *hesitant fuzzy strongly UP-ideal* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x))$, and

(2) $(\forall x, y, z \in A)(\mathrm{h}_{\mathrm{H}}(x) \supseteq \mathrm{h}_{\mathrm{H}}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_{\mathrm{H}}(y))$.

Mosrijai et al. also proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

Theorem 4.1.5 *[25] A hesitant fuzzy set* H *on A is a hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A.*

Theorem 4.1.6 *A hesitant fuzzy set* H *is a constant hesitant fuzzy set on A if and only if the complement of* H *is a constant hesitant fuzzy set on A.*

Proof. Let H be a constant hesitant fuzzy set on A. Then $\mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(0)$ for all $x \in A$. Thus $[0,1] - \mathrm{h}_{\mathrm{H}}(x) = [0,1] - \mathrm{h}_{\mathrm{H}}(0)$ for all $x \in A$. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(0)$ for all $x \in A$. Hence, $\overline{\mathrm{H}}$ is a constant hesitant fuzzy set on A.

Conversely, let $\overline{\mathrm{H}}$ be a constant hesitant fuzzy set on A. Then $\mathrm{h}_{\overline{\mathrm{H}}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(0)$ for all $x \in A$. Thus $[0,1] - \mathrm{h}_{\mathrm{H}}(x) = [0,1] - \mathrm{h}_{\mathrm{H}}(0)$ for all $x \in A$. Therefore, $\mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(0)$ for all $x \in A$. Hence, H is a constant hesitant fuzzy set on A. □

Proposition 4.1.7 *Let* H *be a hesitant fuzzy UP-filter (and also hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of* A. *Then it satisfies the condition:*

$$(\forall x, y \in A)(x \leq y \Rightarrow \mathrm{h}_{\mathrm{H}}(x) \subseteq \mathrm{h}_{\mathrm{H}}(y) \subseteq \mathrm{h}_{\mathrm{H}}(x \cdot y)). \tag{4.1.1}$$

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$, so

$$\mathrm{h}_{\mathrm{H}}(y) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(0) \cap \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(x).$$

By Proposition 3.1.7 (5), we have $y \leq x \cdot y$ and thus $\mathrm{h}_{\mathrm{H}}(y) \subseteq \mathrm{h}_{\mathrm{H}}(x \cdot y)$. □

4.2 Anti-type of hesitant fuzzy sets

In this subsection, we introduce the notions of anti-hesitant fuzzy UP-subalgebras, anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals and anti-hesitant fuzzy strongly UP-ideals of UP-algebras, provide the necessary examples and prove its generalizations.

Definition 4.2.1 A hesitant fuzzy set H on A is called an *anti-hesitant fuzzy UP-subalgebra* of A if it satisfies the following property:

$$(\forall x, y \in A)(\mathrm{h}_{\mathrm{H}}(x \cdot y) \subseteq \mathrm{h}_{\mathrm{H}}(x) \cup \mathrm{h}_{\mathrm{H}}(y)).$$

By Proposition 3.1.7 (1), we have for any $x \in A$,

$$\mathrm{h}_{\mathrm{H}}(0) = \mathrm{h}_{\mathrm{H}}(x \cdot x) \subseteq \mathrm{h}_{\mathrm{H}}(x) \cup \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(x).$$

Example 4.2.2 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = \emptyset, \mathrm{h_H}(1) = \{0.5\}, \mathrm{h_H}(2) = \{0.6\}, \text{ and } \mathrm{h_H}(3) = [0.5, 0.6].$$

Using this data, we can show that H is an anti-hesitant fuzzy UP-subalgebra of A.

Definition 4.2.3 A hesitant fuzzy set H on A is called an *anti-hesitant fuzzy UP-filter* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and

(2) $(\forall x, y \in A)(\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x))$.

Example 4.2.4 Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = \{0.8\}, \mathrm{h_H}(1) = [0.8, 0.9), \mathrm{h_H}(2) = [0.8, 0.9], \mathrm{h_H}(3) = [0.6, 0.9], \text{ and } \mathrm{h_H}(4) = [0.6, 0.9].$$

Using this data, we can show that H is an anti-hesitant fuzzy UP-filter of A.

Definition 4.2.5 A hesitant fuzzy set H on A is called an *anti-hesitant fuzzy UP-ideal* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and

(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y))$.

Example 4.2.6 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = \{1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \{0, 1\}, \text{ and } \mathrm{h_H}(3) = [0, 1].$$

Using this data, we can show that H is an anti-hesitant fuzzy UP-ideal of A.

Definition 4.2.7 A hesitant fuzzy set H on A is called an *anti-hesitant fuzzy strongly UP-ideal* of A if it satisfies the following properties:

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and

(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y))$.

Example 4.2.8 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	0
2	0	3	0	0
3	0	3	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = \{0, 0.2\}, \mathrm{h_H}(1) = \{0, 0.2\}, \mathrm{h_H}(2) = \{0, 0.2\}, \text{ and } \mathrm{h_H}(3) = \{0, 0.2\}.$$

Using this data, we can show that H is an anti-hesitant fuzzy strongly UP-ideal of A.

Theorem 4.2.9 *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if it is a constant hesitant fuzzy set on* A.

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. Then $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and $\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y)$ for all $x, y, z \in A$. For any $x \in A$, we choose $z = x$ and $y = 0$. Then

$$\begin{aligned}
\mathrm{h_H}(x) &\subseteq \mathrm{h_H}((x \cdot 0) \cdot (x \cdot x)) \cup \mathrm{h_H}(0) \\
&= \mathrm{h_H}(0 \cdot 0) \cup \mathrm{h_H}(0) && \text{((UP-3) and Proposition 3.1.7 (1))} \\
&= \mathrm{h_H}(0) \cup \mathrm{h_H}(0) && \text{((UP-2))} \\
&= \mathrm{h_H}(0) \\
&\subseteq \mathrm{h_H}(x),
\end{aligned}$$

so $\mathrm{h_H}(0) = \mathrm{h_H}(x)$. Hence, H is a constant hesitant fuzzy set on A.

Conversely, assume that H is a constant hesitant fuzzy set on A. Then, for any $x \in A, \mathrm{h}_{\mathrm{H}}(0) = \mathrm{h}_{\mathrm{H}}(x)$, so $\mathrm{h}_{\mathrm{H}}(0) \subseteq \mathrm{h}_{\mathrm{H}}(x)$. For any $x, y, z \in A, \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}((z \cdot y) \cdot (z \cdot x)) = \mathrm{h}_{\mathrm{H}}(y)$, so $\mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\mathrm{H}}(y)$. Thus $\mathrm{h}_{\mathrm{H}}(x) \subseteq \mathrm{h}_{\mathrm{H}}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\mathrm{H}}(y)$. Hence, H is an anti-hesitant fuzzy strongly UP-ideal of A. □

Corollary 4.2.10 *For UP-algebras, we can conclude that the notions of anti-hesitant fuzzy strongly UP-ideals and hesitant fuzzy strongly UP-ideals coincide.*

Proof. It is straightforward by Theorems 4.1.5 and 4.2.9. □

Corollary 4.2.11 *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if* $\overline{\mathrm{H}}$ *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A.

Proof. It is straightforward by Theorems 4.1.6 and 4.2.9. □

By using Corollaries 4.2.10 and 4.2.11, we can show that a hesitant fuzzy set H on A is a hesitant fuzzy strongly UP-ideal of A if and only if $\overline{\mathrm{H}}$ on A is a hesitant fuzzy strongly UP-ideal of A.

Theorem 4.2.12 *Every anti-hesitant fuzzy UP-filter of* A *is an anti-hesitant fuzzy UP-subalgebra of* A.

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Then for any $x, y \in A$,

$$\begin{aligned}
\mathrm{h}_{\mathrm{H}}(x \cdot y) &\subseteq \mathrm{h}_{\mathrm{H}}(y \cdot (x \cdot y)) \cup \mathrm{h}_{\mathrm{H}}(y) && \text{(Definition 4.2.3 (2))} \\
&= \mathrm{h}_{\mathrm{H}}(0) \cup \mathrm{h}_{\mathrm{H}}(y) && \text{(Proposition 3.1.7 (5))} \\
&= \mathrm{h}_{\mathrm{H}}(y) && \text{(Definition 4.2.3 (1))} \\
&\subseteq \mathrm{h}_{\mathrm{H}}(x) \cup \mathrm{h}_{\mathrm{H}}(y).
\end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-subalgebra of A. □

The converse of Theorem 4.2.12 is not true in general. By Example 4.2.2, we obtain H is an anti-hesitant fuzzy UP-subalgebra of A. Since $\mathrm{h_H}(1) = \{0.5\} \nsubseteq \{0.6\} = \emptyset \cup \{0.6\} = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(2 \cdot 1) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-filter of A. Therefore, the notion of anti-hesitant fuzzy UP-subalgebras of UP-algebras is generalization of anti-hesitant fuzzy UP-filters.

Theorem 4.2.13 *Every anti-hesitant fuzzy UP-ideal of A is an anti-hesitant fuzzy UP-filter of A.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Then for any $x, y \in A$, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and

$$\begin{aligned} \mathrm{h_H}(y) &= \mathrm{h_H}(0 \cdot y) && \text{((UP-2))} \\ &\subseteq \mathrm{h_H}(0 \cdot (x \cdot y)) \cup \mathrm{h_H}(x) && \text{(Definition 4.2.5 (2))} \\ &= \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x). && \text{((UP-2))} \end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-filter of A. □

The converse of Theorem 4.2.13 is not true in general. By Example 4.2.4, we obtain H is an anti-hesitant fuzzy UP-filter of A. Since $\mathrm{h_H}(3 \cdot 4) = \mathrm{h_H}(3) = [0.6, 0.9] \nsubseteq [0.8, 0.9) = \{0.8\} \cup [0.8, 0.9) = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(3 \cdot (2 \cdot 4)) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-ideal of A. Therefore, the notion of anti-hesitant fuzzy UP-filters of UP-algebras is generalization of anti-hesitant fuzzy UP-ideals.

Theorem 4.2.14 *Every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal of A.*

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. Then for any $x, y \in A$, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and

$$\begin{aligned}
\mathrm{h_H}(x \cdot z) &\subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap \mathrm{h_H}(y) && \text{(Definition 4.2.7 (2))} \\
&= \mathrm{h_H}((z \cdot y) \cdot 0) \cap \mathrm{h_H}(y) && \text{(Proposition 3.1.7 (5))} \\
&= \mathrm{h_H}(0) \cap \mathrm{h_H}(y) && \text{((UP-3))} \\
&= \mathrm{h_H}(y) && \text{(Definition 4.2.7 (1))} \\
&= \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y).
\end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-ideal of A. □

The converse of Theorem 4.2.14 is not true in general. By Theorem 4.2.9, we obtain an anti-hesitant fuzzy strongly UP-ideal is a constant hesitant fuzzy set. But anti-hesitant fuzzy UP-ideal is not a constant hesitant fuzzy set in general. Therefore, the notion of anti-hesitant fuzzy UP-ideals of UP-algebras is generalization of anti-hesitant fuzzy strongly UP-ideals.

From the results of this subsection, we have Figure 1 that is the diagram of anti-type of hesitant fuzzy sets on UP-algebras.

Proposition 4.2.15 *Let* H *be an anti-hesitant fuzzy UP-filter (and also anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of A. Then it satisfies the condition:*

$$(\forall x, y \in A)(x \leq y \Rightarrow \mathrm{h_H}(x) \supseteq \mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y)). \qquad (4.2.1)$$

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$. Since H is an anti-hesitant fuzzy UP-filter (resp., anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of A, we have

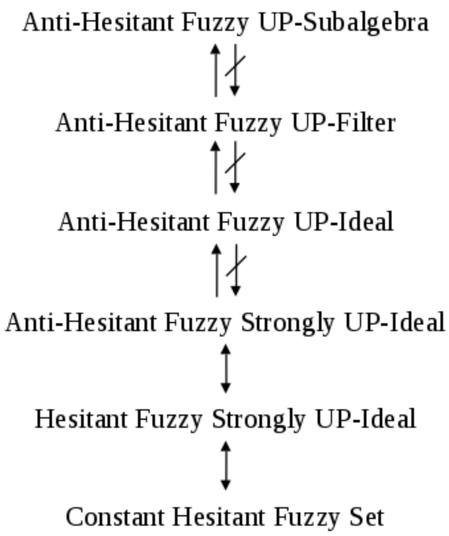

Figure 1: Anti-type of hesitant fuzzy sets on UP-algebras

$$\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) = \mathrm{h_H}(0) \cup \mathrm{h_H}(x) = \mathrm{h_H}(x).$$

By Proposition 3.1.7 (5), we obtain $y \leq x \cdot y$ and thus $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y)$. □

4.3 Level Subsets of a Hesitant Fuzzy Set

The concepts of upper ε-level subsets, upper ε-strong level subsets, lower ε-level subsets, lower ε-strong level subsets, and equal ε-level subsets of a hesitant fuzzy sets was introduced by Akarachai et al. [33] in 2017.

Definition 4.3.1 Let H be a hesitant fuzzy set on A. For any $\varepsilon \in \mathcal{P}([0,1])$, the sets

$$U(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \supseteq \varepsilon\} \text{ and } U^{+}(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \supset \varepsilon\}$$

are called an *upper ε-level subset* and an *upper ε-strong level subset* of H, respectively. The sets

$$L(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \subseteq \varepsilon\} \text{ and } L^{-}(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \subset \varepsilon\}$$

are called a *lower ε-level subset* and a *lower ε-strong level subset* of H, respectively. The set

$$E(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h}_{\mathrm{H}}(x) = \varepsilon\}$$

is called an *equal ε-level subset* of H. Then

$$U(\mathrm{H};\varepsilon) = U^{+}(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon) \text{ and } L(\mathrm{H};\varepsilon) = L^{-}(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon).$$

Lemma 4.3.2 *[33] Let H be a hesitant fuzzy set on A. Then the following statements hold: for any $x, y \in A$,*

(1) $[0,1] - (\mathrm{h}_{\mathrm{H}}(x) \cup \mathrm{h}_{\mathrm{H}}(y)) = ([0,1] - \mathrm{h}_{\mathrm{H}}(x)) \cap ([0,1] - \mathrm{h}_{\mathrm{H}}(y))$, *and*

(2) $[0,1] - (\mathrm{h}_{\mathrm{H}}(x) \cap \mathrm{h}_{\mathrm{H}}(y)) = ([0,1] - \mathrm{h}_{\mathrm{H}}(x)) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y))$.

Proposition 4.3.3 *Let H be a hesitant fuzzy set on A and let $\varepsilon \in \mathcal{P}([0,1])$. Then the following statements hold:*

(1) $U(\mathrm{H};\varepsilon) = L(\overline{\mathrm{H}};[0,1]-\varepsilon)$,

(2) $U^{+}(\mathrm{H};\varepsilon) = L^{-}(\overline{\mathrm{H}};[0,1]-\varepsilon)$,

(3) $L(\mathrm{H};\varepsilon) = U(\overline{\mathrm{H}};[0,1]-\varepsilon)$, *and*

(4) $L^{-}(\mathrm{H};\varepsilon) = U^{+}(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

Proof. (1) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in U(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$ if and only if $[0,1] - \mathrm{h}_{\mathrm{H}}(x) \subseteq [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \subseteq [0,1] - \varepsilon$ if and only if $x \in L(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $U(\mathrm{H};\varepsilon) = L(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

(2) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in U^{+}(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$ if and only if $[0,1] - \mathrm{h}_{\mathrm{H}}(x) \subset [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \subset [0,1] - \varepsilon$ if and only if $x \in L^{-}(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $U^{+}(\mathrm{H};\varepsilon) = L^{-}(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

(3) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in L(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \subseteq \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \supseteq [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \supseteq [0,1] - \varepsilon$ if and only if $x \in U(\overline{\mathrm{H}}; [0,1] - \varepsilon)$. Therefore, $L(\mathrm{H};\varepsilon) = U(\overline{\mathrm{H}}; [0,1] - \varepsilon)$.

(4) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in L^-(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \subset \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \supset [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \supset [0,1] - \varepsilon$ if and only if $x \in U^+(\overline{\mathrm{H}}; [0,1] - \varepsilon)$. Therefore, $L^-(\mathrm{H};\varepsilon) = U^+(\overline{\mathrm{H}}; [0,1] - \varepsilon)$. □

We will discuss the relationships between anti-type of hesitant fuzzy sets on UP-algebras and lower ε-level subsets (resp., lower ε-strong level subsets, upper ε level subsets, upper ε-strong level subsets, equal ε-level subsets) of hesitant fuzzy sets on UP-algebras.

Theorem 4.3.4 *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A.

Proof. Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$ and $y \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \subseteq \varepsilon$ and thus $x \cdot y \in L(\mathrm{H};\varepsilon)$. Hence, $L(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Then $\mathrm{h_H}(x), \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Thus $x, y \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, $L(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A and thus $x \cdot y \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon = \mathrm{h_H}(x) \cup \mathrm{h_H}(y)$. Hence, H is an anti-hesitant fuzzy UP-subalgebra of A. □

Theorem 4.3.5 *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-filter* *of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-filter of A.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $0 \in L(\mathrm{H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in L(\mathrm{H};\varepsilon)$ and $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $y \in L(\mathrm{H};\varepsilon)$. Hence, $L(\mathrm{H};\varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Then $\mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \in L(\mathrm{H};\varepsilon)$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0 \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \subseteq \varepsilon = \mathrm{h_H}(x)$.

Next, let $x, y \in A$. Then $\mathrm{h_H}(x \cdot y), \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \cdot y, x \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(y) \subseteq \varepsilon = \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$. Hence, H is an anti-hesitant fuzzy UP-filter of A. □

Theorem 4.3.6 *A hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $0 \in L(\mathrm{H}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(\mathrm{H}; \varepsilon)$ and $y \in L(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \subseteq \varepsilon$ and thus $x \cdot z \in L(\mathrm{H}; \varepsilon)$. Hence, $L(\mathrm{H}; \varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Then $\mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \in L(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $0 \in L(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h_H}(0) \subseteq \varepsilon = \mathrm{h_H}(x)$.

Next, let $x, y, z \in A$. Then $\mathrm{h_H}(x \cdot (y \cdot z)), \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Thus $x \cdot (y \cdot z), y \in L(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot z) \subseteq \varepsilon = \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Hence, H is an anti-hesitant fuzzy UP-ideal of A. □

Theorem 4.3.7 *Let* H *be a hesitant fuzzy set on A. Then the following statements are equivalent*:

(1) H *is an anti-hesitant fuzzy strongly UP-ideal of A,*

(2) *a nonempty subset $L(\mathrm{H}; \varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and*

(3) *a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$.*

Proof. (1)⇒(2) Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 4.2.9, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H}; \varepsilon) \neq \emptyset$. There exists $a \in L(\mathrm{H};\varepsilon)$ be such that $\mathrm{h_H}(a) \subseteq \varepsilon$. Thus $\mathrm{h_H}(x) = \mathrm{h_H}(a) \subseteq \varepsilon$ for all $x \in A$ and so $x \in L(\mathrm{H};\varepsilon)$ for all $x \in A$. Therefore, $L(\mathrm{H};\varepsilon) = A$. Hence, $L(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(2)⇒(3) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H};\varepsilon) \neq \emptyset$. If $U(\mathrm{H};\varepsilon) \neq A$, then there exist $x \in U(\mathrm{H};\varepsilon)$ and $y \notin U(\mathrm{H};\varepsilon)$. So $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \not\supseteq \varepsilon$. Consider, $\varepsilon_y = \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $y \in L(\mathrm{H};\varepsilon_y)$ and $\varepsilon_y \not\supseteq \varepsilon$. By assumption, we have $L(\mathrm{H};\varepsilon_y)$ is a strongly UP-ideal of A and so $L(\mathrm{H};\varepsilon_y) = A$. Thus $\mathrm{h_H}(x) \subseteq \varepsilon_y$. Since $\mathrm{h_H}(x) \supseteq \varepsilon$, we have $\varepsilon_y \supseteq \varepsilon$, a contradiction. Therefore, $U(\mathrm{H};\varepsilon) = A$. Hence, $U(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(3)⇒(1) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Assume that H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $\mathrm{h_H}(x) \neq \mathrm{h_H}(y)$. Now, $x \in U(\mathrm{H};\mathrm{h_H}(x)) \neq \emptyset$ and $y \in U(\mathrm{H};\mathrm{h_H}(y)) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\mathrm{h_H}(x))$ and $U(\mathrm{H};\mathrm{h_H}(y))$ are strongly UP-ideals of A and thus $U(\mathrm{H};\mathrm{h_H}(x)) = A = U(\mathrm{H};\mathrm{h_H}(y))$. Then $x \in U(\mathrm{H};\mathrm{h_H}(y))$ and $y \in U(\mathrm{H};\mathrm{h_H}(x))$. Thus $\mathrm{h_H}(x) \supseteq \mathrm{h_H}(y)$ and $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x)$. So $\mathrm{h_H}(x) = \mathrm{h_H}(y)$, a contradiction. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we obtain H is an anti-hesitant fuzzy strongly UP-ideal of A. □

Theorem 4.3.8 *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* H *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^-(\mathrm{H};\varepsilon)$ *is a UP-subalgebra of* A *if* $L^-(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A*, then* H *is an anti-hesitant fuzzy UP-subalgebra of* A*.*

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^{-}(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in L^{-}(\mathrm{H};\varepsilon)$ and $y \in L^{-}(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \subset \varepsilon$ and thus $x \cdot y \in L^{-}(\mathrm{H};\varepsilon)$. Hence, $L^{-}(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^{-}(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$ such that $\mathrm{h_H}(x \cdot y) \nsubseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y)$. Since Im(H) is a chain, we have $\mathrm{h_H}(x \cdot y) \supset \mathrm{h_H}(x) \cup \mathrm{h_H}(y)$. Choose $\varepsilon = \mathrm{h_H}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Thus $x, y \in L^{-}(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L^{-}(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in L^{-}(\mathrm{H};\varepsilon)$. Thus $\mathrm{h_H}(x \cdot y) \subset \varepsilon = \mathrm{h_H}(x \cdot y)$, a contradiction. Therefore, $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y)$ for all $x, y \in A$. Hence, H is an anti-hesitant fuzzy UP-subalgebra of A. □

Example 4.3.9 Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	0
4	0	2	2	4	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = (0,1), \mathrm{h_H}(1) = [0,1), \mathrm{h_H}(2) = (0,1], \mathrm{h_H}(3) = [0,1), \text{ and } \mathrm{h_H}(4) = [0,1].$$

Then Im(H) is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^{-}(\mathrm{H};\varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^{-}(\mathrm{H};\varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then. $L^{-}(\mathrm{H};\varepsilon) = \{0,1,2,3\}$. Using this data, we can show that all nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Since $\mathrm{h_H}(3\cdot 1)=\mathrm{h_H}(2)=(0,1]\nsubseteq[0,1)=\mathrm{h_H}(3)\cup\mathrm{h_H}(1)$, we have H is not an anti-hesitant fuzzy UP-subalgebra of A.

Theorem 4.3.10 *Let* H *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy UP-filter of* A, *then for all* $\varepsilon\in\mathcal{P}([0,1])$, $L^-(\mathrm{H};\varepsilon)$ *is a UP-filter of* A *if* $L^-(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* Im(H) *is a chain and for all* $\varepsilon\in\mathcal{P}([0,1])$, *a nonempty subset* $L^-(\mathrm{H};\varepsilon)$ *of* A *is a UP-filter of* A, *then* H *is an anti-hesitant fuzzy UP-filter of* A.

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon\in\mathcal{P}([0,1])$ be such that $L^-(\mathrm{H};\varepsilon)\neq\emptyset$ and let $x\in A$ be such that $x\in L^-(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x)\subset\varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(0)\subseteq\mathrm{h_H}(x)\subset\varepsilon$ and thus $0\in L^-(\mathrm{H};\varepsilon)$.

Next, let $x,y\in A$ be such that $x\cdot y\in L^-(\mathrm{H};\varepsilon)$ and $x\in L^-(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x\cdot y)\subset\varepsilon$ and $\mathrm{h_H}(x)\subset\varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y)\subseteq\mathrm{h_H}(x\cdot y)\cup\mathrm{h_H}(x)\subset\varepsilon$ and thus $y\in L^-(\mathrm{H};\varepsilon)$. Hence, $L^-(\mathrm{H};\varepsilon)$ is a UP-filter of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon\in\mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Assume that there exists $x\in A$ such that $\mathrm{h_H}(0)\nsubseteq\mathrm{h_H}(x)$. Since Im(H) is a chain, we have $\mathrm{h_H}(0)\supset\mathrm{h_H}(x)$. Choose $\varepsilon=\mathrm{h_H}(0)\in\mathcal{P}([0,1])$. Then $\mathrm{h_H}(x)\subset\mathrm{h_H}(0)=\varepsilon$. Thus $x\in L^-(\mathrm{H};\varepsilon)\neq\emptyset$. By assumption, we have $L^-(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0\in L^-(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0)\subset\varepsilon=\mathrm{h_H}(0)$, a contradiction. Hence, $\mathrm{h_H}(0)\subseteq\mathrm{h_H}(x)$ for all $x\in A$.

Next, assume that there exist $x,y\in A$ such that $\mathrm{h_H}(y)\nsubseteq\mathrm{h_H}(x\cdot y)\cup\mathrm{h_H}(x)$. Since Im(H) is a chain, we have $\mathrm{h_H}(y)\supset\mathrm{h_H}(x\cdot y)\cup\mathrm{h_H}(x)$. Choose $\varepsilon=\mathrm{h_H}(y)\in$

$\mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot y) \subset \varepsilon$ and $\mathrm{h_H}(x) \subset \varepsilon$. Thus $x \cdot y, x \in L^-(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in L^-(\mathrm{H};\varepsilon)$. Thus $\mathrm{h_H}(y) \subset \varepsilon = \mathrm{h_H}(y)$, a contradiction. Therefore, $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$ for all $x, y \in A$. Hence, H is an anti-hesitant fuzzy UP-filter of A. □

Example 4.3.11 Let $A = \{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = (0,1), \mathrm{h_H}(1) = [0,1), \mathrm{h_H}(2) = (0,1], \mathrm{h_H}(3) = [0,1], \text{ and } \mathrm{h_H}(4) = [0,1].$$

Then Im(H) is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^-(\mathrm{H};\varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^-(\mathrm{H};\varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then $L^-(\mathrm{H};\varepsilon) = \{0,1,2\}$. Using this data, we can show that all nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Since $\mathrm{h_H}(2) = (0,1] \nsubseteq [0,1) = \mathrm{h_H}(1) \cup \mathrm{h_H}(1) = \mathrm{h_H}(1 \cdot 2) \cup \mathrm{h_H}(1)$, we have H is not an anti-hesitant fuzzy UP-filter of A.

Theorem 4.3.12 *Let* H *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy UP-ideal of* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $L^-(\mathrm{H};\varepsilon)$ *is a UP-ideal of* A *if* $L^-(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{H};\varepsilon)$ *of* A *is a UP-ideal of* A*, then* H *is an anti-hesitant fuzzy UP-ideal of* A*.*

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(\mathrm{H};\varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L^-(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subset \varepsilon$ and thus $0 \in L^-(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(\mathrm{H};\varepsilon)$ and $y \in L^-(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \subset \varepsilon$ and thus $x \cdot z \in L^-(\mathrm{H};\varepsilon)$. Hence, $L^-(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h_H}(0) \nsubseteq \mathrm{h_H}(x)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h_H}(0) \supset \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subset \mathrm{h_H}(0) = \varepsilon$. Thus $x \in L^-(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $0 \in L^-(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \subset \varepsilon = \mathrm{h_H}(0)$, a contradiction. Hence, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ for all $x \in A$.

Next, assume that there exist $x, y, z \in A$ such that $\mathrm{h_H}(x \cdot z) \nsubseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h_H}(x \cdot z) \supset \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Choose $\varepsilon = \mathrm{h_H}(x \cdot z) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Thus $x \cdot (y \cdot z), y \in L^-(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L^-(\mathrm{H};\varepsilon)$. Thus $\mathrm{h_H}(x \cdot z) \subset \varepsilon = \mathrm{h_H}(x \cdot z)$, a contradiction. Therefore, $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$ for all $x, y, z \in A$. Hence, H is an anti-hesitant fuzzy UP-ideal of A. □

Example 4.3.13 Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = (0,1), \mathrm{h_H}(1) = [0,1), \mathrm{h_H}(2) = [0,1], \mathrm{h_H}(3) = (0,1], \text{ and } \mathrm{h_H}(4) = [0,1].$$

Then Im(H) is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^{-}(\mathrm{H};\varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^{-}(\mathrm{H};\varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then $L^{-}(\mathrm{H};\varepsilon) = \{0,1,3\}$. Using this data, we can show that all nonempty subset $L^{-}(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A. Since $\mathrm{h_H}(0 \cdot 1) = \mathrm{h_H}(1) = [0,1) \nsubseteq (0,1] = \mathrm{h_H}(0) \cup \mathrm{h_H}(3) = \mathrm{h_H}(0 \cdot (3 \cdot 1)) \cup \mathrm{h_H}(3)$, we have H is not an anti-hesitant fuzzy UP-ideal of A.

Theorem 4.3.14 *Let* H *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^{-}(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A if $L^{-}(\mathrm{H};\varepsilon)$ is nonempty, and*

(2) *if* Im(H) *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^{-}(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A, then* H *is an anti-hesitant fuzzy strongly UP-ideal of A.*

Proof. (1) Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 4.2.9, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) =$

$h_H(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(H;\varepsilon) \neq \emptyset$. There exists $a \in L^-(H;\varepsilon)$ be such that $h_H(a) \subset \varepsilon$. Thus $h_H(x) = h_H(a) \subset \varepsilon$ for all $x \in A$ and so $x \in L^-(H;\varepsilon)$ for all $x \in A$. Therefore, $L^-(H;\varepsilon) = A$. Hence, $L^-(H;\varepsilon)$ is a strongly UP-ideal of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(H;\varepsilon)$ of A is a strongly UP-ideal of A. Assume that H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $h_H(x) \neq h_H(y)$. Since Im(H) is a chain, we have $h_H(x) \subset h_H(y)$ or $h_H(x) \supset h_H(y)$. Without loss of generality, assume that $h_H(x) \subset h_H(y)$, then $x \in L^-(H; h_H(y)) \neq \emptyset$. By assumption, we have $L^-(H; h_H(y))$ is a strongly UP-ideal of A and so $L^-(H; h_H(y)) = A$. Thus $y \in A = L^-(H; h_H(y))$ and so $h_H(y) \subset h_H(y)$, a contradiction. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we obtain H is an anti-hesitant fuzzy strongly UP-ideal of A. □

Example 4.3.15 Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$h_H(0) = (0,1], \text{ and } h_H(1) = [0,1).$$

Then Im(H) is not a chain. If $\varepsilon \subseteq [0,1)$ or $\varepsilon \subseteq (0,1]$, then $L^-(H;\varepsilon) = \emptyset$. If $\varepsilon = [0,1]$, then $L^-(H;\varepsilon) = A$. Thus a nonempty subset $L^-(H;\varepsilon)$ of A is a strongly UP-ideal of A. By Theorem 4.2.9 and H is not a constant hesitant fuzzy set on A, we have H is not an anti-hesitant fuzzy strongly UP-ideal of A.

Theorem 4.3.16 *A hesitant fuzzy set* $\overline{H}$ *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U(H;\varepsilon)$ *of*

A is a UP-subalgebra of A.

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in U(\mathrm{H};\varepsilon)$ and $y \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h_H}(x \cdot y) \subseteq ([0,1] - \mathrm{h_H}(x)) \cup ([0,1] - \mathrm{h_H}(y)) = [0,1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \supseteq \varepsilon$. Therefore, $x \cdot y \in U(\mathrm{H};\varepsilon)$. Hence, $U(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Choose $\varepsilon = \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \supseteq \varepsilon$. Thus $x, y \in U(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in U(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot y) \supseteq \varepsilon = \mathrm{h_H}(x) \cap \mathrm{h_H}(y)$. By Lemma 4.3.2 (2), we have

$$\begin{aligned}
\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) &= [0,1] - \mathrm{h_H}(x \cdot y) \\
&\subseteq [0,1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y)) \\
&= ([0,1] - \mathrm{h_H}(x)) \cup ([0,1] - \mathrm{h_H}(y)) \\
&= \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y).
\end{aligned}$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. □

Theorem 4.3.17 *A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-filter of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-filter of A.*

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus

$[0,1]-\mathrm{h_H}(0) \subseteq [0,1]-\mathrm{h_H}(x)$. Therefore, $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) \supseteq \varepsilon$. Hence, $0 \in U(\mathrm{h_H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in U(\mathrm{H};\varepsilon)$ and $x \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h_H}(x) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h_H}(y) \subseteq ([0,1] - \mathrm{h_H}(x \cdot y)) \cup ([0,1] - \mathrm{h_H}(x)) = [0,1] - (\mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x))$. Thus $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x) \supseteq \varepsilon$. Therefore, $y \in U(\mathrm{H};\varepsilon)$. Hence, $U(\mathrm{H};\varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0 \in U(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \supseteq \varepsilon = \mathrm{h_H}(x)$. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) = [0,1] - \mathrm{h_H}(0) \subseteq [0,1] - \mathrm{h_H}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(x)$.

Next, let $x, y \in A$. Choose $\varepsilon = \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h_H}(x) \supseteq \varepsilon$. Thus $x \cdot y, x \in U(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in U(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(y) \supseteq \varepsilon = \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x)$. By Lemma 4.3.2 (2), we have

$$\begin{aligned}
\mathrm{h}_{\overline{\mathrm{H}}}(y) &= [0,1] - \mathrm{h_H}(y) \\
&\subseteq [0,1] - (\mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x)) \\
&= ([0,1] - \mathrm{h_H}(x \cdot y)) \cup ([0,1] - \mathrm{h_H}(x)) \\
&= \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x).
\end{aligned}$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. □

Theorem 4.3.18 *A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A.*

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1]-\mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1]-\mathrm{h}_{\mathrm{H}}(x)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Hence, $0 \in U(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(\mathrm{H};\varepsilon)$ and $y \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z) \subseteq ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y)) = [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y))$. Thus $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Therefore, $x \cdot z \in U(\mathrm{H};\varepsilon)$. Hence, $U(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $0 \in U(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x)$. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) = [0,1] - \mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1] - \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(x)$.

Next, let $x, y, z \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Thus $x \cdot (y \cdot z), y \in U(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $x \cdot z \in U(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y)$. By Lemma 4.3.2 (2), we have

$$\begin{aligned}
\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) &= [0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z) \\
&\subseteq [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y)) \\
&= ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y)) \\
&= \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y).
\end{aligned}$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. □

Theorem 4.3.19 *Let* H *be a hesitant fuzzy set on* A*. Then the following state-*

ments are equivalent:

(1) $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy strongly UP-ideal of* A,

(2) *a nonempty subset* $U(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$, *and*

(3) *a nonempty subset* $L(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$.

Proof. It is straightforward by Theorem 4.3.7 and Corollary 4.2.11. □

Theorem 4.3.20 *Let* H *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-subalgebra of* A, *then for all* $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{H};\varepsilon)$ *is a UP-subalgebra of* A *if* $U^+(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$, *a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A, *then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-subalgebra of* A.

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in U^+(\mathrm{H};\varepsilon)$ and $y \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h_H}(x \cdot y) \subseteq ([0,1] - \mathrm{h_H}(x)) \cup ([0,1] - \mathrm{h_H}(y)) = [0,1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \supset \varepsilon$. Therefore, $x \cdot y \in U^+(\mathrm{H};\varepsilon)$. Hence, $U^+(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$

such that $h_{\overline{H}}(x \cdot y) \nsubseteq h_{\overline{H}}(x) \cup h_{\overline{H}}(y)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $h_{\overline{H}}(x \cdot y) \supset h_{\overline{H}}(x) \cup h_{\overline{H}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - h_{H}(x \cdot y) \supset ([0,1] - h_{H}(x)) \cup ([0,1] - h_{H}(y)) = [0,1] - (h_{H}(x) \cap h_{H}(y))$. Thus $h_{H}(x \cdot y) \subset h_{H}(x) \cap h_{H}(y)$. Choose $\varepsilon = h_{H}(x \cdot y) \in \mathcal{P}([0,1])$. Then $h_{H}(x) \supset \varepsilon$ and $h_{H}(y) \supset \varepsilon$. Thus $x, y \in U^{+}(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U^{+}(\mathrm{H}; \varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in U^{+}(\mathrm{H}; \varepsilon)$. Thus $h_{H}(x \cdot y) \supset \varepsilon = h_{H}(x \cdot y)$, a contradiction. Therefore, $h_{\overline{H}}(x \cdot y) \subseteq h_{\overline{H}}(x) \cup h_{\overline{H}}(y)$ for all $x, y \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. □

Example 4.3.21 From a UP-algebra $A = \{0, 1, 2, 3, 4\}$ of Example 4.3.9. We define a hesitant fuzzy set H on A as follows:

$$h_{H}(0) = \{0, 1\}, h_{H}(1) = \{1\}, h_{H}(2) = \{0\}, h_{H}(3) = \{1\}, \text{ and } h_{H}(4) = \emptyset.$$

Then $\mathrm{Im}(\mathrm{H})$ is not a chain. If $\varepsilon = \{1\}$ or $\varepsilon = \{0\}$, then $U^{+}(\mathrm{H}; \varepsilon) = \{0\}$. If $\varepsilon = \emptyset$, then $U^{+}(\mathrm{H}; \varepsilon) = \{0, 1, 3\}$. Otherwise, $U^{+}(\mathrm{H}; \varepsilon) = \emptyset$. Using this data, we can show that all nonempty subset $U^{+}(\mathrm{H}; \varepsilon)$ of A is a UP-subalgebra of A. By Definition 4.1.2, we have

$$h_{\overline{H}}(0) = (0, 1), h_{\overline{H}}(1) = [0, 1), h_{\overline{H}}(2) = (0, 1], h_{\overline{H}}(3) = [0, 1), \text{ and } h_{\overline{H}}(3) = [0, 1].$$

Since $h_{\overline{H}}(3 \cdot 1) = h_{\overline{H}}(2) = (0, 1] \nsubseteq [0, 1) = h_{\overline{H}}(3) \cup h_{\overline{H}}(1)$, we have H is not an anti-hesitant fuzzy UP-subalgebra of A.

Theorem 4.3.22 *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-filter of* A*, then for all* $\varepsilon \in \mathcal{P}([0, 1])$*,* $U^{+}(\mathrm{H}; \varepsilon)$ *is a UP-filter of* A *if* $U^{+}(\mathrm{H}; \varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $U^{+}(\mathrm{H}; \varepsilon)$ *of* A *is a UP-filter of* A*, then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-filter of* A*.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^{+}(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U^{+}(\mathrm{H};\varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1]-\mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1]-\mathrm{h}_{\mathrm{H}}(x)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Hence, $0 \in U^{+}(\mathrm{h}_{\mathrm{H}};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in U^{+}(\mathrm{H};\varepsilon)$ and $x \in U^{+}(\mathrm{H};\varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot y) \supset \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 4.3.2 (2), we have $[0,1]-\mathrm{h}_{\mathrm{H}}(y) \subseteq ([0,1]-\mathrm{h}_{\mathrm{H}}(x \cdot y)) \cup ([0,1]-\mathrm{h}_{\mathrm{H}}(x)) = [0,1]-(\mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x))$. Thus $\mathrm{h}_{\mathrm{H}}(y) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Therefore, $y \in U^{+}(\mathrm{H};\varepsilon)$. Hence, $U^{+}(\mathrm{H};\varepsilon)$ is a UP-filter of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Assume that there exists $x \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(0) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x)$. and thus $[0,1]-\mathrm{h}_{\mathrm{H}}(0) \supset [0,1]-\mathrm{h}_{\mathrm{H}}(x)$. So $\mathrm{h}_{\mathrm{H}}(0) \subset \mathrm{h}_{\mathrm{H}}(x)$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Thus $x \in U^{+}(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^{+}(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0 \in L^{-}(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supset \varepsilon = \mathrm{h}_{\mathrm{H}}(0)$, a contradiction. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for all $x \in A$.

Next, assume that there exist $x, y \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(y) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 4.3.2 (2), we have $[0,1]-\mathrm{h}_{\mathrm{H}}(y) \supset ([0,1]-\mathrm{h}_{\mathrm{H}}(x \cdot y)) \cup ([0,1]-\mathrm{h}_{\mathrm{H}}(x)) = [0,1]-(\mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x))$. Thus $\mathrm{h}_{\mathrm{H}}(y) \subset \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x)$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot y) \supset \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Thus $x \cdot y, x \in U^{+}(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^{+}(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in U^{+}(\mathrm{H};\varepsilon)$. Thus $\mathrm{h}_{\mathrm{H}}(y) \supset \varepsilon = \mathrm{h}_{\mathrm{H}}(y)$, a contradiction. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for all $x, y \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. □

Example 4.3.23 From a UP-algebra $A = \{0, 1, 2, 3, 4\}$ of Example 4.3.11. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = \{0,1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \{0\}, \mathrm{h_H}(3) = \emptyset, \text{ and } \mathrm{h_H}(4) = \emptyset.$$

Then Im(H) is not a chain. If $\varepsilon = \{1\}$ or $\varepsilon = \{0\}$, then $U^+(\mathrm{H};\varepsilon) = \{0\}$. If $\varepsilon = \emptyset$, then $U^+(\mathrm{H};\varepsilon) = \{0,1,2\}$. Otherwise, $U^+(\mathrm{H};\varepsilon) = \emptyset$. Using this data, we can show that all nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. By Definition 4.1.2, we have

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1), \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1), \mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1], \mathrm{h}_{\overline{\mathrm{H}}}(3) = [0,1], \text{ and } \mathrm{h}_{\overline{\mathrm{H}}}(4) = [0,1].$$

Since $\mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1] \not\subseteq [0,1) = \mathrm{h}_{\overline{\mathrm{H}}}(1) \cup \mathrm{h}_{\overline{\mathrm{H}}}(1) = \mathrm{h}_{\overline{\mathrm{H}}}(1 \cdot 2) \cup \mathrm{h}_{\overline{\mathrm{H}}}(1)$, we have $\overline{\mathrm{H}}$ is not an anti-hesitant fuzzy UP-filter of A.

Theorem 4.3.24 *Let* H *be a hesitant fuzzy set on A. Then the following statements hold:*

(1) *if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^+(\mathrm{H};\varepsilon)$ is a UP-ideal of A if $U^+(\mathrm{H};\varepsilon)$ is nonempty, and*

(2) *if* Im(H) *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1] - \mathrm{h_H}(0) \subseteq [0,1] - \mathrm{h_H}(x)$. Therefore, $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) \supset \varepsilon$. Hence, $0 \in U^+(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(\mathrm{H};\varepsilon)$ and $y \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy

UP-ideal of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z) \subseteq ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y)) = [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y))$. Thus $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y) \supset \varepsilon$. Therefore, $x \cdot z \in U^+(\mathrm{H}; \varepsilon)$. Hence, $U^+(\mathrm{H}; \varepsilon)$ is a UP-ideal of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(0) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Then $[0,1] - \mathrm{h}_{\mathrm{H}}(0) \supset [0,1] - \mathrm{h}_{\mathrm{H}}(x)$. Thus $\mathrm{h}_{\mathrm{H}}(0) \subset \mathrm{h}_{\mathrm{H}}(x)$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x) \supset \varepsilon$. Thus $x \in U^+(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $0 \in U^+(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supset \varepsilon = \mathrm{h}_{\mathrm{H}}(0)$, a contradiction. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for any $x \in A$.

Next, assume that there exist $x, y, z \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 4.3.2 (2), we have $[0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z) \supset ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y)) = [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y))$. Thus $\mathrm{h}_{\mathrm{H}}(x \cdot z) \subset \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y)$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot z) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \supset \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(y) \supset \varepsilon$. Thus $x \cdot (y \cdot z), y \in U^+(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L^-(\mathrm{H}; \varepsilon)$. Thus, $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supset \varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot z)$, a contradiction. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$ for all $x, y, z \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. □

Example 4.3.25 From a UP-algebra $A = \{0, 1, 2, 3, 4\}$ of Example 4.3.13. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h}_{\mathrm{H}}(0) = \{0, 1\}, \mathrm{h}_{\mathrm{H}}(1) = \{1\}, \mathrm{h}_{\mathrm{H}}(2) = \emptyset, \mathrm{h}_{\mathrm{H}}(3) = \{0\}, \text{ and } \mathrm{h}_{\mathrm{H}}(4) = \emptyset.$$

Then $\mathrm{Im}(\mathrm{H})$ is not a chain. If $\varepsilon = \{1\}$ or $\varepsilon = \{0\}$, then $U^+(\mathrm{H}; \varepsilon) = \{0\}$. If $\varepsilon = \emptyset$, then $U^+(\mathrm{H}; \varepsilon) = \{0, 1, 3\}$. Otherwise, $U^+(\mathrm{H}; \varepsilon) = \emptyset$. Using this data, we can show that all nonempty subset $U^+(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. By Definition

4.1.2, we have

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1), \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1), \mathrm{h}_{\overline{\mathrm{H}}}(2) = [0,1], \mathrm{h}_{\overline{\mathrm{H}}}(3) = (0,1], \text{ and } \mathrm{h}_{\overline{\mathrm{H}}}(4) = [0,1].$$

Since $\mathrm{h}_{\overline{\mathrm{H}}}(0 \cdot 1) = \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1) \nsubseteq (0,1] = \mathrm{h}_{\overline{\mathrm{H}}}(0) \cup \mathrm{h}_{\overline{\mathrm{H}}}(3) = \mathrm{h}_{\overline{\mathrm{H}}}(0 \cdot (3 \cdot 1)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(3)$, we have $\overline{\mathrm{H}}$ is not an anti-hesitant fuzzy UP-ideal of A.

Theorem 4.3.26 *Let* H *be a hesitant fuzzy set on A. Then the following statements hold:*

(1) *if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^{+}(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A if $U^{+}(\mathrm{H};\varepsilon)$ is nonempty, and*

(2) *if* Im(H) *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 4.2.9, we obtain $\overline{\mathrm{H}}$ is a constant hesitant fuzzy set on A. By Corollary 4.2.11, we have H is a constant hesitant fuzzy set on A and so $\mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^{+}(\mathrm{H};\varepsilon) \neq \emptyset$. There exists $a \in U^{+}(\mathrm{H};\varepsilon)$ be such that $\mathrm{h}_{\mathrm{H}}(a) \supset \varepsilon$. Thus $\mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\mathrm{H}}(a) \supset \varepsilon$ for all $x \in A$ and so $x \in U^{+}(\mathrm{H};\varepsilon)$ for all $x \in A$. Therefore, $U^{+}(\mathrm{H};\varepsilon) = A$. Hence, $U^{+}(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Assume that $\overline{\mathrm{H}}$ is not a constant hesitant fuzzy set on A. By Corollary 4.2.11, we have H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $\mathrm{h}_{\mathrm{H}}(x) \neq \mathrm{h}_{\mathrm{H}}(y)$. Since Im(H) is a chain, we have $\mathrm{h}_{\mathrm{H}}(x) \subset \mathrm{h}_{\mathrm{H}}(y)$ or $\mathrm{h}_{\mathrm{H}}(x) \supset \mathrm{h}_{\mathrm{H}}(y)$. Without loss of generality,

assume that $h_H(x) \subset h_H(y)$, then $y \in U^+(H; h_H(x)) \neq \emptyset$. By assumption, we have $U^+(H; h_H(x))$ is a strongly UP-ideal of A and so $U^+(H; h_H(x)) = A$. Thus $x \in A = U^+(H; h_H(x))$ and so $h_H(x) \subset h_H(x)$, a contradiction. Therefore, $\overline{H}$ is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we obtain $\overline{H}$ is an anti-hesitant fuzzy strongly UP-ideal of A. □

Example 4.3.27 From a UP-algebra $A = \{0, 1\}$ of Example 4.3.15. We define a hesitant fuzzy set H on A as follows:

$$h_H(0) = \{0\}, \text{ and } h_H(1) = \{1\}.$$

Then Im(H) is not a chain. If $\varepsilon = \emptyset$, then $U^+(H; \varepsilon) = A$. Otherwise, $U^+(H; \varepsilon) = \emptyset$. Thus a nonempty subset $U^+(H; \varepsilon)$ of A is a strongly UP-ideal of A. By Definition 4.1.2, we have

$$h_{\overline{H}}(0) = (0, 1], \text{ and } h_{\overline{H}}(1) = [0, 1).$$

By Theorem 4.2.9 and because $\overline{H}$ is not a constant hesitant fuzzy set on A, we have $\overline{H}$ is not an anti-hesitant fuzzy strongly UP-ideal of A.

Theorem 4.3.28 *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $E(H; \varepsilon)$ *of* A *is a UP-subalgebra of* A *where* $L^-(H; \varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $E(H; \varepsilon) \neq \emptyset$ but $L^-(H; \varepsilon) = \emptyset$, and let $x, y \in A$ be such that $x \in E(H; \varepsilon)$ and $y \in E(H; \varepsilon)$. Then $h_H(x) = \varepsilon$ and $h_H(y) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-subalgebra of A, we have $h_H(x \cdot y) \subseteq h_H(x) \cup h_H(y) = \varepsilon$. Thus $x \cdot y \in L(H; \varepsilon)$. Since $L^-(H; \varepsilon)$ is empty, we obtain $L(H; \varepsilon) = L^-(H; \varepsilon) \cup E(H; \varepsilon) = \emptyset \cup E(H; \varepsilon) = E(H; \varepsilon)$. Therefore, $x \cdot y \in E(H; \varepsilon)$. Hence, $E(H; \varepsilon)$ is a UP-subalgebra of A. □

The following example show that the converse of Theorem 4.3.28 is not true in general.

Example 4.3.29 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$h_{\mathrm{H}}(0) = \emptyset, h_{\mathrm{H}}(1) = [0, 0.6], h_{\mathrm{H}}(2) = [0, 0.3], \text{ and } h_{\mathrm{H}}(3) = [0, 0.3].$$

If $\varepsilon \neq \emptyset$, then $L^{-}(\mathrm{H}; \varepsilon) \neq \emptyset$. If $\varepsilon = \emptyset$, then $L^{-}(\mathrm{H}; \varepsilon) = \emptyset$ and $E(\mathrm{H}; \varepsilon) = \{0\}$. Thus $E(\mathrm{H}; \varepsilon)$ is clearly a UP-subalgebra of A. Since $h_{\mathrm{H}}(3 \cdot 2) = h_{\mathrm{H}}(1) = [0, 0.6] \nsubseteq [0, 0.3] = h_{\mathrm{H}}(3) \cup h_{\mathrm{H}}(2)$, we have H is not an anti-hesitant fuzzy UP-subalgebra of A.

Theorem 4.3.30 *If a hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $E(\mathrm{H}; \varepsilon)$ of A is a UP-filter of A where $L^{-}(\mathrm{H}; \varepsilon)$ is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $E(\mathrm{H}; \varepsilon) \neq \emptyset$ but $L^{-}(\mathrm{H}; \varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H}; \varepsilon)$. Then $h_{\mathrm{H}}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we obtain $h_{\mathrm{H}}(0) \subseteq h_{\mathrm{H}}(x) = \varepsilon$ and thus $0 \in L(\mathrm{H}; \varepsilon)$. Since $L^{-}(\mathrm{H}; \varepsilon)$ is empty, we have $0 \in L(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in E(\mathrm{H}; \varepsilon)$ and $x \in E(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x \cdot y) = \varepsilon$ and $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) = \varepsilon$. Thus $y \in L(\mathrm{H}; \varepsilon)$. Since $L^{-}(\mathrm{H}; \varepsilon)$ is empty, we obtain $L(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$. Therefore, $y \in E(\mathrm{H}; \varepsilon)$. Hence, $E(\mathrm{H}; \varepsilon)$ is a UP-filter of A. □

The converse of Theorem 4.3.30 is not true in general. By Example 4.3.29, we still have $E(\mathrm{H}; \varepsilon) = \{0\}$ is a UP-filter of A. Since $\mathrm{h_H}(1) = [0, 0.6] \nsubseteq [0, 0.3] = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(2 \cdot 1) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-filter of A.

Theorem 4.3.31 *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-ideal of* A*, then* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $E(\mathrm{H}; \varepsilon)$ *of* A *is a UP-ideal of* A *where* $L^{-}(\mathrm{H}; \varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $E(\mathrm{H}; \varepsilon) \neq \emptyset$ but $L^{-}(\mathrm{H}; \varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-ideal of A, we obtain $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in L(\mathrm{H}; \varepsilon)$. Since $L^{-}(\mathrm{H}; \varepsilon)$ is empty, we have $0 \in L(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in E(\mathrm{H}; \varepsilon)$ and $y \in E(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot z \in L(\mathrm{H}; \varepsilon)$. Since $L^{-}(\mathrm{H}; \varepsilon)$ is empty, we obtain $L(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$. Therefore, $x \cdot z \in E(\mathrm{H}; \varepsilon)$. Hence, $E(\mathrm{H}; \varepsilon)$ is a UP-ideal of A. □

The converse of Theorem 4.3.31 is not true in general. By Example 4.3.29, we still have $E(\mathrm{H}; \varepsilon) = \{0\}$ is a UP-ideal of A. Since $\mathrm{h_H}(0 \cdot 1) = \mathrm{h_H}(1) =$

$[0, 0.6] \nsubseteq [0, 0.3] = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(0 \cdot (2 \cdot 1)) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-ideal of A.

Theorem 4.3.32 *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if* $E(\mathrm{H}; \mathrm{h_H}(0))$ *is a strongly UP-ideal of* A.

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 4.2.9, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Then $E(\mathrm{H}; \mathrm{h_H}(0)) = A$. Hence, $E(\mathrm{H}; \mathrm{h_H}(0))$ is a strongly UP-ideal of A.

Conversely, assume that $E(\mathrm{H}; \mathrm{h_H}(0))$ is a strongly UP-ideal of A. Then $E(\mathrm{H}; \mathrm{h_H}(0)) = A$ and so $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 4.2.9, H is an anti-hesitant fuzzy strongly UP-ideal of A. $\square$

Moreover, we still obtain theorems of equal ε-level subsets with a hesitant fuzzy UP-subalgebra. (resp., hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal)

Theorem 4.3.33 *If a hesitant fuzzy set* H *on* A *is a hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $E(\mathrm{H}; \varepsilon)$ *of* A *is a UP-subalgebra of* A *where* $U^+(\mathrm{H}; \varepsilon)$ *is empty.*

Proof. Assume that H is a hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $E(\mathrm{H}; \varepsilon) \neq \emptyset$ but $U^+(\mathrm{H}; \varepsilon) = \emptyset$, and let $x, y \in A$ be such that $x \in E(\mathrm{H}; \varepsilon)$ and $y \in E(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Since H is a hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot y \in U(\mathrm{H}; \varepsilon)$. Since $U^+(\mathrm{H}; \varepsilon)$ is empty, we obtain $U(\mathrm{H}; \varepsilon) = U^+(\mathrm{H}; \varepsilon) \cup E(\mathrm{H}; \varepsilon) = \emptyset \cup E(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$. Therefore, $x \cdot y \in E(\mathrm{H}; \varepsilon)$. Hence, $E(\mathrm{H}; \varepsilon)$ is a UP-subalgebra of A. $\square$

The following example show that the converse of Theorem 4.3.33 is not true in general.

Example 4.3.34 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	0
3	0	0	1	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H on A as follows:

$$\mathrm{h_H}(0) = [0, 1], \mathrm{h_H}(1) = \{0\}, \mathrm{h_H}(2) = [0, 0.1], \text{ and } \mathrm{h_H}(3) = [0, 0.1].$$

If $\varepsilon \neq [0, 1]$, then $U^+(\mathrm{H}; \varepsilon) \neq \emptyset$. If $\varepsilon = [0, 1]$, then $U^+(\mathrm{H}; \varepsilon) = \emptyset$ and $E(\mathrm{H}; \varepsilon) = \{0\}$. Thus $E(\mathrm{H}; \varepsilon)$ is clearly a UP-subalgebra of A. Since $\mathrm{h_H}(3 \cdot 2) = \mathrm{h_H}(1) = \{0\} \not\supseteq [0, 0.1] = \mathrm{h_H}(3) \cap \mathrm{h_H}(2)$, we have H is not a hesitant fuzzy UP-subalgebra of A.

Theorem 4.3.35 *If a hesitant fuzzy set* H *on* A *is a hesitant fuzzy UP-filter of* A*, then for all* $\varepsilon \in \mathcal{P}([0, 1])$*, a nonempty subset* $E(\mathrm{H}; \varepsilon)$ *of* A *is a UP-filter of* A *where* $U^+(\mathrm{H}; \varepsilon)$ *is empty.*

Proof. Assume that H is a hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $E(\mathrm{H}; \varepsilon) \neq \emptyset$ but $U^+(\mathrm{H}; \varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is a hesitant fuzzy UP-filter of A, we obtain $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in U(\mathrm{H}; \varepsilon)$. Since $U^+(\mathrm{H}; \varepsilon)$ is empty, we have $0 \in U(\mathrm{H}; \varepsilon) = E(\mathrm{H}; \varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in E(\mathrm{H};\varepsilon)$ and $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) = \varepsilon$ and $\mathrm{h_H}(x) = \varepsilon$. Because H is a hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x) = \varepsilon$. Thus $y \in L(\mathrm{H};\varepsilon)$. Since $U^+(\mathrm{H};\varepsilon)$ is empty, we obtain $U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $y \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-filter of A. □

The converse of Theorem 4.3.35 is not true in general. By Example 4.3.34, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-filter of A. Since $\mathrm{h_H}(1) = \{0\} \not\supseteq [0, 0.1] = \mathrm{h_H}(0) \cap \mathrm{h_H}(3) = \mathrm{h_H}(3 \cdot 1) \cap \mathrm{h_H}(3)$, we have H is not a hesitant fuzzy UP-filter of A.

Theorem 4.3.36 *If a hesitant fuzzy set* H *on* A *is a hesitant fuzzy UP-ideal of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $E(\mathrm{H};\varepsilon)$ *of* A *is a UP-ideal of* A *where* $U^+(\mathrm{H};\varepsilon)$ *is empty.*

Proof. Assume that H is a hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $U^+(\mathrm{H};\varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is a hesitant fuzzy UP-filter of A, we obtain $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in U(\mathrm{H};\varepsilon)$. Since $U^+(\mathrm{H};\varepsilon)$ is empty, we have $0 \in U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in E(\mathrm{H};\varepsilon)$ and $y \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Since H is a hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot z \in U(\mathrm{H};\varepsilon)$. Since $L^-(\mathrm{H};\varepsilon)$ is empty, we obtain $U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $x \cdot z \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-ideal of A. □

The converse of Theorem 4.3.36 is not true in general. By Example 4.3.34, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-ideal of A. Since $\mathrm{h_H}(3 \cdot 2) = \mathrm{h_H}(1) =$

$\{0\} \not\supseteq [0, 0.1] = \mathrm{h_H}(0) \cap \mathrm{h_H}(2) = \mathrm{h_H}(3 \cdot (2 \cdot 2)) \cap \mathrm{h_H}(2)$, we have H is not a hesitant fuzzy UP-ideal of A.

Theorem 4.3.37 *A hesitant fuzzy set* H *on A is a hesitant fuzzy strongly UP-ideal of A if and only if $E(\mathrm{H}; \mathrm{h_H}(0))$ is a strongly UP-ideal of A.*

Proof. It is straightforward by Theorem 4.3.32 and 4.2.9. □

CHAPTER V

HESITANT FUZZY SOFT SETS

5.1 Hesitant fuzzy soft sets

The concept of hesitant fuzzy soft sets, which is a link between classical soft sets and hesitant fuzzy sets was introduced by Babitha and John [5] in 2013.

Definition 5.1.1 Let X be a reference set (or an initial universe set) and P be a set of parameters. Let $\mathrm{HFS}(X)$ be the set of all hesitant fuzzy sets on X and Y be a nonempty subset of P. A pair $(\widetilde{\mathrm{H}}, Y)$ is called a *hesitant fuzzy soft set* over X where $\widetilde{\mathrm{H}}$ is a mapping given by

$$\widetilde{\mathrm{H}}\colon Y \to \mathrm{HFS}(X), p \mapsto \widetilde{\mathrm{H}}[p].$$

We will apply the hesitant fuzzy soft set theory to UP-algebras.

Definition 5.1.2 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *hesitant fuzzy soft UP-subalgebra* based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-subalgebra*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-subalgebra of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *hesitant fuzzy soft UP-subalgebra* of A.

Theorem 5.1.3 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A, then it satisfies the property:*

$$(\forall p \in Y \forall x \in A)(\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.1.1}$$

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A and let $p \in Y$ and $x \in A$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. □

Example 5.1.4 Let $(\mathcal{P}_{\emptyset}(\{a, b\}), \cdot, \emptyset)$ is the power UP-algebra of type 1 which a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	$\emptyset$	$\{a\}$	$\{b\}$	X
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	X
$\{a\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{b\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
X	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_{\emptyset}(\{a, b\})$ by the following table:

$\widetilde{\mathrm{H}}$	$\emptyset$	$\{a\}$	$\{b\}$	X
p_1	$\{0.3, 0.4\}$	$\{0.3\}$	$\{0.4\}$	$\emptyset$
p_2	$[0.6, 0.9]$	$\{0.9\}$	$[0.6, 0.9]$	$[0.6, 0.9]$
p_3	$(0.3, 0.8)$	$[0.3, 0.5]$	$\{0.4, 0.5\}$	$\{0.4, 0.5\}$
p_4	$[0, 1)$	$[0, 1)$	$[0, 1)$	$[0, 1)$

Then $(\widetilde{\mathrm{H}}, Y)$ satisfies the property (5.1.1), but not a hesitant fuzzy soft UP-subalgebra of A based on parameter p_2. Indeed,

$$\begin{aligned}
\mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{b\} \cdot X) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{a\}) = \{0.9\} &\not\supseteq [0.6, 0.9] \\
&= [0.6, 0.9] \cap [0.6, 0.9] \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{b\}) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(X).
\end{aligned}$$

Theorem 5.1.5 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the* *condition:*

$$(\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.1.2}$$

Then it is a hesitant fuzzy soft UP-subalgebra of A.

Proof. Let $p \in Y$ and $x, y \in A$. By Proposition 3.1.7 (5) and (UP-3), we have $x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0$ and thus $x \leq y \cdot (x \cdot y)$. It follows form (5.1.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-subalgebra of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A. □

Corollary 5.1.6 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (5.1.2)*, then it satisfies the property* (5.1.1).

Proof. It is straightforward form Theorems 5.1.5 and 5.1.3. □

Theorem 5.1.7 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A and N is a nonempty subset of Y, then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of A.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A and $\emptyset \neq N \subseteq Y$. Then $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A for all $p \in Y$. Since $N \subseteq Y$, we have $(\widetilde{\mathrm{H}}|_N, N)$ is a p-hesitant fuzzy soft UP-subalgebra of A for all $p \in N$. Therefore, $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of A. □

The following example shows that there exists a nonempty subset N of Y such that $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of A, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft UP-subalgebra of A.

Example 5.1.8 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4, p_5\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A as the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3	4
p_1	$\{0.5, 0.6, 0.7\}$	$\{0.5\}$	$\{0.6\}$	$\{0.6\}$	$\emptyset$
p_2	$[0.4, 0.6]$	$(0.4, 0.6)$	$(0.5, 0.6)$	$(0.5, 0.55)$	$\{0.5\}$
p_3	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2\}$	$\{0.1, 0.2\}$	$\{0.2\}$	$\{0.2\}$
p_4	$[0.7, 1)$	$[0.7, 1]$	$\{0.7\}$	$\{0.5, 0.7\}$	$[0.5, 0.7]$
p_5	$\{0.9\}$	$\{0.9\}$	$\{0.9\}$	$\{0.9\}$	$\{0.9\}$

Then $\widetilde{\mathrm{H}}[p_4]$ is not a hesitant fuzzy UP-subalgebra of A. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1 \cdot 1) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0) = [0.7, 1) \not\supseteq [0.7, 1] \\ = [0.7, 1] \cap [0.7, 1] \\ = \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1). \end{aligned}$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft UP-subalgebra of A. But if we choose $N = \{p_1, p_2, p_3, p_5\}$, then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of A.

Definition 5.1.9 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *hesitant fuzzy soft UP-filter* based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-filter*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-filter of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-filter of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *hesitant fuzzy soft UP-filter* of A.

From [25], we know that every hesitant fuzzy UP-filter of A is a hesitant fuzzy UP-subalgebra. Then we have the following Theorem:

Theorem 5.1.10 *Every p-hesitant fuzzy soft UP-filter of A is a p-hesitant fuzzy soft UP-subalgebra.*

The following example shows that the converse of Theorem 5.1.10 is not true in general.

Example 5.1.11 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 4.2.2. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$\{0.5, 0.6\}$	$\{0.5\}$	$\{0.6\}$	$\emptyset$
p_2	$[0.4, 0.6]$	$(0.4, 0.6)$	$[0.4, 0.6]$	$[0.4, 0.6]$
p_3	$(0.2, 0.7)$	$[0.3, 0.5]$	$\{0.4, 0.5\}$	$\{0.5\}$
p_4	$\{0.8\}$	$\{0.8\}$	$\{0.8\}$	$\{0.8\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A, but not a hesitant fuzzy soft UP-filter of A based on parameter p_1. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1) = \{0.5\} &\not\supseteq \{0.6\} \\ &= \{0.5, 0.6\} \cap \{0.6\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2 \cdot 1) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2). \end{aligned}$$

Theorem 5.1.12 *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is a hesitant fuzzy soft UP-filter of* A *if and only if it satisfies the condition* (5.1.2)*.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A. Let $p \in Y$ and let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filter of A. By Proposition 4.1.7, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y)$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.1.2). Let $p \in Y$ and let $x \in A$. By Corollary 5.1.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y \in A$. By Proposition 3.1.7 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq x \cdot y$. It follows from (5.1.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filer of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-filter of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A. □

Theorem 5.1.13 *Let* $(\widetilde{\mathrm{H}}, Y)$ *be a hesitant fuzzy soft set over* A *which satisfies* *the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.1.3)$$

Then it is a hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1.3). Let $p \in Y$ and let $x, y \in A$. By Proposition 3.1.7 (1), we have $0 \cdot ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0$ and thus $0 \leq (x \cdot y) \cdot (x \cdot y)$. It follows form (5.1.3) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A. □

Corollary 5.1.14 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (5.1.3)*, then it satisfies the condition* (5.1.2)*.*

Proof. It is straightforward form Theorems 5.1.13 and 5.1.12. □

Theorem 5.1.15 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.1.4)$$

Then it is a hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1.4). Let $p \in Y$ and let $x, y \in A$. By Proposition 3.1.7 (1) and (UP-2), we have $(x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq 0 \cdot (x \cdot y)$.

It follows form (5.1.4) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A. □

Corollary 5.1.16 *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over* A *which satisfies the condition* (5.1.4)*, then it satisfies the condition* (5.1.2)*.*

Proof. It is straightforward form Theorems 5.1.15 and 5.1.12. □

Definition 5.1.17 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *hesitant fuzzy soft UP-ideal* based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *hesitant fuzzy soft UP-ideal* of A.

From [25], we know that every hesitant fuzzy UP-ideal of A is a hesitant fuzzy UP-filter. Then we have the following Theorem:

Theorem 5.1.18 *Every p-hesitant fuzzy soft UP-ideal of A is a p-hesitant fuzzy soft UP-filter.*

The following example shows that the converse of Theorem 5.1.18 is not true in general.

Example 5.1.19 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$[0.2, 0.5]$	$\{0.5\}$	$\emptyset$	$\{0.3, 0.4\}$
p_2	$[0.4, 0.6]$	$(0.4, 0.6)$	$\{0.5\}$	$\{0.5\}$
p_3	$[0.5, 0.7]$	$\{0.5, 0.6\}$	$\{0.5, 0.6\}$	$[0.5, 0.6)$
p_4	$\{0.1, 0.6\}$	$\{0.1, 0.6\}$	$\{0.1, 0.6\}$	$\{0.1, 0.6\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A, but not a hesitant fuzzy soft UP-ideal of A based on parameter p_1. Indeed,

$$\begin{aligned}
\mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot 2) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) = \emptyset &\not\supseteq \{0.5\} \\
&= [0.2, 0.5] \cap \{0.5\} \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1) \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot (1 \cdot 2)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1).
\end{aligned}$$

Theorem 5.1.20 *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft UP-ideal of* A*, then it satisfies the condition* (5.1.3)*.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $x \leq w\cdot(y\cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of A and $x\cdot(w\cdot(y\cdot z)) = 0$. Thus $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z)) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(w\cdot(y\cdot z)))\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0)\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)$. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z))\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)$. □

Example 5.1.21 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$[0.2, 0.5]$	$\{0.5\}$	$\emptyset$	$\{0.3, 0.4\}$
p_2	$[0.4, 0.6]$	$(0.4, 0.6)$	$\{0.5\}$	$\{0.5\}$
p_3	$[0.5, 0.7]$	$[0.5, 0.7)$	$\emptyset$	$\{0.7\}$
p_4	$\{0.7, 0.8, 0.9\}$	$\{0.7, 0.8\}$	$\{0.8\}$	$\{0.8\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of A, but does not satisfy the condition (5.1.3).

Theorem 5.1.22 *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is a hesitant fuzzy soft UP-ideal of* A *if and only if it satisfies the condition* (5.1.4).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A. Let $p \in Y$ and $w, x, y, z \in A$ be such that $w \leq x\cdot(y\cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of A. By Proposition 4.1.7, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z))$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z))\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.1.4). Let $p \in Y$ and let $x \in A$. By Corollaries 5.1.16 and 5.1.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. By Proposition 3.1.7 (1), we have $(x\cdot(y\cdot z))\cdot(x\cdot(y\cdot z)) = 0$ and thus $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows form (5.1.4) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A. □

Theorem 5.1.23 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.1.5)$$

Then it is a hesitant fuzzy soft UP-ideal of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1.5). Let $p \in Y$ and let $x, y, z \in A$. By Proposition 3.1.7 (1) and (UP-3), we have $(x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))) = (x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot 0) = (x \cdot (y \cdot z)) \cdot 0 = 0$ and thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))$. It follows form (5.1.5) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A. □

Corollary 5.1.24 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (5.1.5)*, then it satisfies the conditions* (5.1.3) *and* (5.1.4)*.*

Proof. It is straightforward form Theorems 5.1.23, 5.1.20, and 5.1.22. □

Definition 5.1.25 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *hesitant fuzzy soft strongly UP-ideal* based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft strongly UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy strongly UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft strongly UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *hesitant fuzzy soft strongly UP-ideal* of A.

From [25], we know that every hesitant fuzzy strongly UP-ideal of A is a hesitant fuzzy UP-ideal. Then we have the following Theorem:

Theorem 5.1.26 *Every p-hesitant fuzzy soft strongly UP-ideal of A is a p-hesitant fuzzy soft UP-ideal.*

The following example shows that the converse of Theorem 5.1.26 is not true in general.

Example 5.1.27 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$[0.2, 0.5]$	$\{0.4\}$	$(0.2, 0.4)$	$(0.2, 0.4)$
p_2	$[0.9, 1]$	$\{1\}$	$\{1\}$	$\{1\}$
p_3	$[0, 0.2]$	$(0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$
p_4	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A, but not a hesitant fuzzy soft strongly UP-ideal of A based on parameter p_2. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(3) = \{1\} &\not\supseteq [0.9, 1] \\ &= [0.9, 1] \cap [0.9, 1] \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}((1 \cdot 0) \cdot (1 \cdot 3)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0). \end{aligned}$$

By Theorems 5.1.10, 5.1.18, and 5.1.26 and Examples 5.1.11, 5.1.19, and 5.1.27, we have that the notion of p-hesitant fuzzy soft UP-subalgebras is a generalization of p-hesitant fuzzy soft UP-filters, the notion of p-hesitant fuzzy soft UP-filters is a generalization of p-hesitant fuzzy soft UP-ideals, and the notion of p-hesitant fuzzy soft UP-ideals is a generalization of p-hesitant fuzzy soft strongly UP-ideals.

Theorem 5.1.28 *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is a hesitant fuzzy soft strongly UP-ideal of* A *if and only if it satisfies the condition* (5.1.5).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of A. By Proposition 4.1.7, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \subseteq$

$h_{\widetilde{H}[p]}((z \cdot y) \cdot (z \cdot x))$. Therefore,

$$h_{\widetilde{H}[p]}(x) \supseteq h_{\widetilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cap h_{\widetilde{H}[p]}(y) \supseteq h_{\widetilde{H}[p]}(w) \cap h_{\widetilde{H}[p]}(y).$$

Conversely, assume that $(\widetilde{H}, Y)$ satisfies the condition (5.1.5). Let $p \in Y$ and let $x \in A$. By Corollaries 5.1.24, 5.1.16 and 5.1.6, we have $h_{\widetilde{H}[p]}(0) \supseteq h_{\widetilde{H}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. It follows from (5.1.5) that

$$h_{\widetilde{H}[p]}(x) \supseteq h_{\widetilde{H}[p]}((z \cdot y) \cdot (z \cdot x)) \cap h_{\widetilde{H}[p]}(y).$$

Therefore, $\widetilde{H}[p]$ is a hesitant fuzzy strongly UP-ideal of A. Hence, $(\widetilde{H}, Y)$ is a p-hesitant fuzzy soft strongly UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of A. □

Theorem 5.1.29 *Let $(\widetilde{H}, Y)$ is a hesitant fuzzy soft set over A such that $\emptyset \neq N \subseteq Y$. Then the following statements are hold:*

(1) *if $(\widetilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A, then $(\widetilde{H}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A, and*

(2) *there exists $(\widetilde{H}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A, but $(\widetilde{H}, Y)$ is not a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A.*

Proof. (1) Assume that $(\widetilde{H}, Y)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A. In the same way as Theorem 5.1.7, we can show that $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A.

(2) By Example 5.1.27 (resp., Example 5.1.19, Example 5.1.11), if we choose $N = \{p_4\}$ (resp., $\{p_3, p_4\}$, $\{p_3, p_4\}$) , then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of A. □

Definition 5.1.30 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *constant hesitant fuzzy soft set* based on $p \in Y$ (we shortly call a p-*constant hesitant fuzzy soft set*) over A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a constant hesitant fuzzy set on A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-constant hesitant fuzzy soft set over A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *constant hesitant fuzzy soft set* over A.

Theorem 5.1.31 *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of A and let $p \in Y$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of A. By Theorem 4.1.5, we obtain $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. Thus $(\widetilde{\mathrm{H}}, Y)$ is a p-constant hesitant fuzzy soft set over A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A.

Conversely, let $p \in Y$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A. Then $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. By Theorem 4.1.5, we have $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of A. Since p is arbitrary, we state that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of A. □

From the results of this subsection, we have Figure 2 that is the diagram of hesitant fuzzy soft sets over UP-algebras.

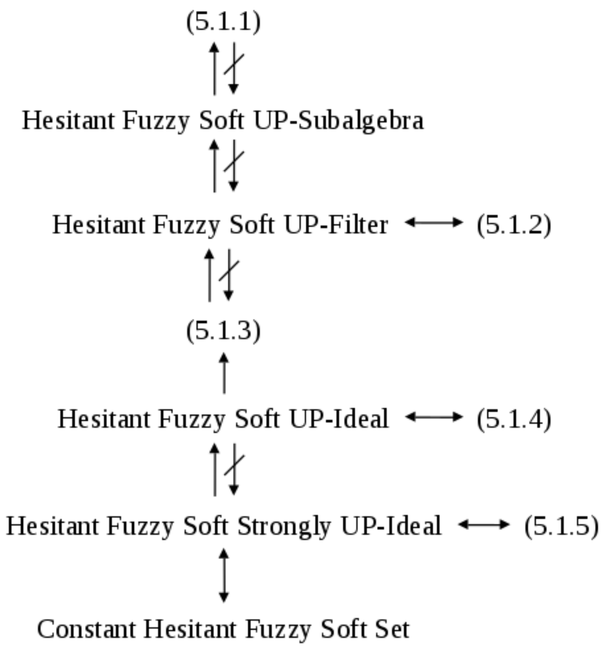

Figure 2: Hesitant fuzzy soft sets over UP-algebras

5.2 Anti-type of hesitant fuzzy soft sets

In this subsection, we introduce the notions of anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy soft strongly UP-ideals of UP-algebras, provide the necessary examples and prove its generalizations.

Definition 5.2.1 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called an *anti-hesitant fuzzy soft UP-subalgebra* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-subalgebra*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-subalgebra of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-subalgebra* of A.

Theorem 5.2.2 *If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, then it satisfies the property:*

$$(\forall p \in Y \forall x \in A)(\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.2.1}$$

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. Let $p \in Y$ and $x \in A$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. □

Example 5.2.3 From the power UP-algebra of type 1 $(\mathcal{P}_{\emptyset}(\{a, b\}), \cdot, \emptyset)$ of Example 5.1.4, and let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_{\emptyset}(\{a, b\})$ by the following table:

$\widetilde{\mathrm{H}}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
p_1	$\emptyset$	$\{0.1\}$	$\{0.2\}$	$\{0.1, 0.2\}$
p_2	$(0.6, 0.7)$	$(0.6, 0.7)$	$[0.6, 0.7]$	$[0.6, 0.7]$
p_3	$\{0.5\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$(0.2, 0.7)$
p_4	$\{0.9\}$	$[0.8, 0.9]$	$\{0.8, 0.9\}$	$\{0.8, 0.9\}$

Then $(\widetilde{\mathrm{H}}, Y)$ satisfies the property (5.2.1), but not an anti-hesitant fuzzy soft UP-subalgebra of $\mathcal{P}_{\emptyset}(\{a, b\})$ based on parameter p_4. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\} \cdot \{a, b\}) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a\}) = [0.8, 0.9] &\nsubseteq \{0.8, 0.9\} \\ &= \{0.8, 0.9\} \cup \{0.8, 0.9\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a, b\}).\end{aligned}$$

Theorem 5.2.4 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the* *condition:*

$$(\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.2.2}$$

Then it is an anti-hesitant fuzzy soft UP-subalgebra of A.

Proof. Let $p \in Y$ and $x, y \in A$. By Proposition 3.1.7 (5) and (UP-3), we have $x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0$ and thus $x \leq y \cdot (x \cdot y)$. It follows from (5.2.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

Corollary 5.2.5 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (5.2.2)*, then it satisfies the property* (5.2.1)*.*

Proof. It is straightforward from Theorems 5.2.4 and 5.2.2. □

Theorem 5.2.6 *If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A and N is a nonempty subset of Y, then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A and $\emptyset \neq N \subseteq Y$. Since $N \subseteq Y$, we have $(\widetilde{\mathrm{H}}|_N, N)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A for all $p \in N$. Therefore, $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

By Example 5.2.3, we have $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A. But if we choose $N = \{p_1, p_2, p_3\}$, then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. We can conclude that there exists a nonempty subset N of Y such that $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A.

Definition 5.2.7 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-filter* based on $p \in Y$ (we shortly call a *anti-p-hesitant fuzzy soft UP-filter*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-filter of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-filter of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-filter* of A.

By Theorem 4.2.12, we know that every anti-hesitant fuzzy UP-filter of A is an anti-hesitant fuzzy UP-subalgebra. Then we have the following Theorem:

Theorem 5.2.8 *Every p-anti-hesitant fuzzy soft UP-filter of A is a p-anti-hesitant fuzzy soft UP-subalgebra.*

The following example shows that the converse of Theorem 5.2.8 is not true in general.

Example 5.2.9 From the power UP-algebra of type 1 $(\mathcal{P}_{\emptyset}(\{a, b\}), \cdot, \emptyset)$ of Example 5.1.4, and let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_{\emptyset}(\{a,b\})$ by the following table:

$\widetilde{\mathrm{H}}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
p_1	$\emptyset$	$\{0\}$	$\{0.1\}$	$\{0, 0.1\}$
p_2	$\{0.2\}$	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$
p_3	$\{0.4\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$[0.4, 0.6)$
p_4	$\{0.8\}$	$[0.7, 0.9]$	$[0.7, 0.9]$	$\{0.8\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, but not an anti-hesitant fuzzy soft UP-filter of $\mathcal{P}_{\emptyset}(\{a,b\})$ based on parameters p_3 and p_4. Indeed,

$$\begin{aligned}
\mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a,b\}) = [0.4, 0.6) &\nsubseteq [0.4, 0.5] \\
&= [0.4, 0.5] \cup \{0.4, 0.5\} \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\}) \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\} \cdot \{a,b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\}), \\
\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\}) = [0.7, 0.9] &\nsubseteq \{0.8\} \\
&= \{0.8\} \cup \{0.8\} \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\emptyset) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\}) \\
&= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\} \cdot \{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\}).
\end{aligned}$$

Theorem 5.2.10 *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is an anti-hesitant fuzzy soft UP-filter of* A *if and only if it satisfies the condition* (5.2.2).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. Let $p \in Y$ and let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-filter of A. By Proposition 4.2.15, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y)$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.2.2). Let $p \in Y$ and let $x \in A$. By Corollary 5.2.5, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y \in A$. By Proposition 3.1.7 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq x \cdot y$. It follows from (5.2.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-filer of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-filter of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Theorem 5.2.11 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.2.3)$$

Then it is an anti-hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2.3). Let $p \in Y$ and let $x, y \in A$. By Proposition 3.1.7 (1) and (UP-2), we have $0 \cdot ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0$ and thus $0 \leq (x \cdot y) \cdot (x \cdot y)$. It follows from (5.2.3) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Corollary 5.2.12 *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (5.2.3)*, then it satisfies the condition* (5.2.2)*.*

Proof. It is straightforward from Theorems 5.2.11 and 5.2.10. □

Example 5.2.13 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.4\}$	$\{0.4\}$
p_2	$\{0\}$	$\{0, 1\}$	$\{0, 1\}$	$\{0\}$
p_3	$\{0.9\}$	$\{0.9\}$	$[0.9, 1]$	$[0.9, 1]$
p_4	$(0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A, But it does not satisfy the condition (5.2.3) because $3 \cdot (0 \cdot (1 \cdot 2)) = 0$ implies

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(3 \cdot 2) &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(2) \\ &= [0.9, 1] \\ &\not\subseteq \{0.9\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(1). \end{aligned}$$

Theorem 5.2.14 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.2.4)$$

Then it is an anti-hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2.4). Let $p \in Y$ and let $x, y \in A$. By Proposition 3.1.7 (1) and (UP-2), we have $(x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq 0 \cdot (x \cdot y)$.

It follows from (5.2.4) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Corollary 5.2.15 *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over* A *which satisfies the condition* (5.2.4)*, then it satisfies the condition* (5.2.2)*.*

Proof. It is straightforward from Theorems 5.2.14 and 5.2.10. □

Definition 5.2.16 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-ideal* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-ideal* of A.

By Theorem 4.2.13, we know that every anti-hesitant fuzzy UP-ideal of A is an anti-hesitant fuzzy UP-filter. Then we have the following Theorem:

Theorem 5.2.17 *Every p-anti-hesitant fuzzy soft UP-ideal of A is a p-anti-hesitant fuzzy soft UP-filter.*

The following example shows that the converse of Theorem 5.2.17 is not true in general.

Example 5.2.18 From a UP-algebra $A = \{0, 1, 2, 3, 4\}$ of Example 4.2.4. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$

over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3	4
p_1	$\{0.7\}$	$[0.7, 0.8)$	$[0.7, 0.9]$	$[0.6, 1]$	$[0.6, 1]$
p_2	$\emptyset$	$\{0.6\}$	$\{0.6\}$	$\{0.6, 0.9\}$	$\{0.6, 0.9\}$
p_3	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$
p_4	$\{0.3\}$	$\{0.3\}$	$[0.2, 0.4]$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A, but not an anti-hesitant fuzzy soft UP-ideal of A based on parameters p_1. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot 4) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3) = [0.7, 0.9] &\not\subseteq [0.7, 0.8) \\ &= \{0.7\} \cup [0.7, 0.8) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot (2 \cdot 4)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2). \end{aligned}$$

Theorem 5.2.19 *If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A, then it satisfies the condition* (5.2.3).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A and $x \cdot (w \cdot (y \cdot z)) = 0$. Thus $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (w \cdot (y \cdot z))) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)$. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)$. $\square$

Theorem 5.2.20 *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft UP-ideal of A if and only if it satisfies the condition* (5.2.4).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. Let

$p \in Y$ and $w, x, y, z \in A$ be such that $w \leq x \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A. By Proposition 4.2.15, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z))$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.2.4). Let $p \in Y$ and let $x \in A$. By Corollaries 5.2.15 and 5.2.5, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. By Proposition 3.1.7 (1), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$ and thus $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (5.2.4) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. □

Theorem 5.2.21 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.2.5)$$

Then it is an anti-hesitant fuzzy soft UP-ideal of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2.5). Let $p \in Y$ and let $x, y, z \in A$. By Proposition 3.1.7 (1) and (UP-3), we have $(x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))) = (x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot 0) = (x \cdot (y \cdot z)) \cdot 0 = 0$ and thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))$. It follows from (5.2.5) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. □

Corollary 5.2.22 *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over* A *which satisfies the condition* (5.2.5), *then it satisfies the conditions* (5.2.3) *and* (5.2.4).

Proof. It is straightforward from Theorems 5.2.21, 5.2.19 and 5.2.20. □

Definition 5.2.23 Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft strongly UP-ideal* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft strongly UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy strongly UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft strongly UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft strongly UP-ideal* of A.

By Theorem 4.2.14, we know that every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal. Then we have the following Theorem:

Theorem 5.2.24 *Every* p*-anti-hesitant fuzzy soft strongly UP-ideal of* A *is a* p*-anti-hesitant fuzzy soft UP-ideal.*

The following example shows that the converse of Theorem 5.2.24 is not true in general.

Example 5.2.25 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$

over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$[0.3, 0.8)$	$[0.3, 0.8)$	$[0.3, 0.8)$	$[0.3, 0.8]$
p_2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
p_3	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$
p_4	$(0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A, but not an anti-hesitant fuzzy soft strongly UP-ideal of A based on parameter p_4. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(2) = [0, 0.1] &\nsubseteq (0, 0.1] \\ &= (0, 0.1] \cup (0, 0.1] \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}((2 \cdot 0) \cdot (2 \cdot 2)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0). \end{aligned}$$

By Theorems 5.2.8, 5.2.17, and 5.2.24 and Examples 5.2.9, 5.2.18, and 5.2.25, we have that the notion of p-anti-hesitant fuzzy soft UP-subalgebras is a generalization of p-anti-hesitant fuzzy soft UP-filters, the notion of p-anti-hesitant fuzzy soft UP-filters is a generalization of p-anti-hesitant fuzzy soft UP-ideals, and the notion of p-anti-hesitant fuzzy soft UP-ideals is a generalization of p-anti-hesitant fuzzy soft strongly UP-ideals.

Theorem 5.2.26 *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is an anti-hesitant fuzzy soft strongly UP-ideal of* A *if and only if it satisfies the condition* (5.2.5).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Proposition 4.2.15, we have

$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x))$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.2.5). Let $p \in Y$ and let $x \in A$. By Corollaries 5.2.22, 5.2.15 and 5.2.5, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. It follows from (5.2.5) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft strongly UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. □

Theorem 5.2.27 *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A and N be a nonempty subset of Y. Then the following statements are hold:*

(1) *if $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A, then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A, and*

(2) *there exists $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A.*

Proof. (1) Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A.

In the same way as Theorem 5.2.6, we can show that $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A.

(2) Example 5.2.25 (resp., Example 5.2.18, Example 5.2.9), if we choose $N = \{p_2, p_3\}$ (resp., $\{p_3, p_4\}$, $\{p_1, p_2\}$) , then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A. □

Theorem 5.2.28 *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A and let $p \in Y$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 4.2.9, we obtain $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. Thus $(\widetilde{\mathrm{H}}, Y)$ is a p-constant hesitant fuzzy soft set over A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A.

Conversely, let $p \in Y$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A. Then $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we have $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. Since p is arbitrary, we state that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. □

Corollary 5.2.29 *For UP-algebras, we can conclude that the notions of anti-hesitant fuzzy soft strongly UP-ideals and hesitant fuzzy soft strongly UP-ideals coincide.*

Proof. It is straightforward by Theorems 5.1.31 and 5.2.28. □

From the results of this subsection, we have Figure 3 that is the diagram of anti-type of hesitant fuzzy soft sets over UP-algebras.

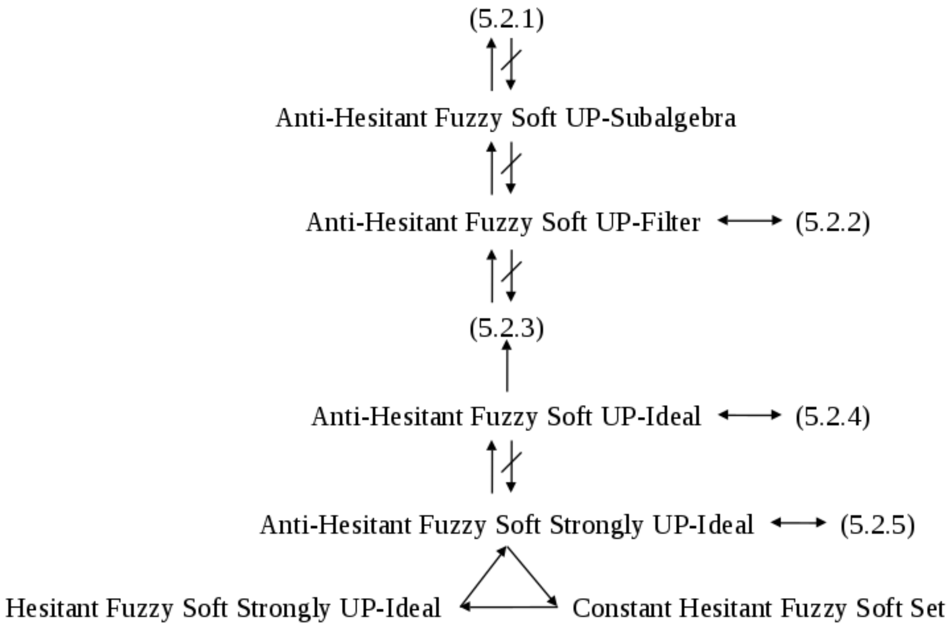

Figure 3: Anti-type of hesitant fuzzy soft sets over UP-algebras

CHAPTER VI

OPERATIONS

6.1 Operations on hesitant fuzzy sets

Torra [40] defined several operations on hesitant fuzzy sets in 2010. For instance, the *intersection* operation $\cap_T$, the *union* operation $\cup_T$ and so on.

Definition 6.1.1 Let H and F be hesitant fuzzy sets on a reference set X. Then the following operations are defined:

(1) $\mathrm{H} \cap_T \mathrm{F} = \{(x, \{h \in (\mathrm{h_H}(x) \cup \mathrm{h_F}(x)) \mid h \leq \min\{\sup \mathrm{h_H}(x), \sup \mathrm{h_F}(x)\}\}) \mid x \in X\}$, and

(2) $\mathrm{H} \cup_T \mathrm{F} = \{(x, \{h \in (\mathrm{h_H}(x) \cup \mathrm{h_F}(x)) \mid h \geq \max\{\inf \mathrm{h_H}(x), \inf \mathrm{h_F}(x)\}\}) \mid x \in X\}$.

We can write the *intersection* and the *union* of hesitant fuzzy sets with different form by

$$\mathrm{H} \cap_T \mathrm{F} = \{(x, \bigcup_{\gamma_1 \in \mathrm{h_H}(x), \gamma_2 \in \mathrm{h_F}(x)} \{\min\{\gamma_1, \gamma_2\}\}) \mid x \in X\}$$

and

$$\mathrm{H} \cup_T \mathrm{F} = \{(x, \bigcup_{\gamma_1 \in \mathrm{h_H}(x), \gamma_2 \in \mathrm{h_F}(x)} \{\max\{\gamma_1, \gamma_2\}\}) \mid x \in X\}.$$

First, we will consider about results of the intersection and the union of hesitant fuzzy sets on UP-algebras.

The following example show that the intersection of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A.

Example 6.1.2 Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0, 0.5, 1\} & \text{if } x = 0, \\ \{0\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.5, 1\} & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \{0, 5\} & \text{if } x = 2, \\ \{0.5, 1\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0, 0.5, 1\} & \text{if } x = 0, \\ \{0\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{0.5, 1\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(1) \\ &= \{0\} \\ &\not\supseteq \{0.5\} \\ &= \{0.5\} \cap \{0.5, 1\} \end{aligned}$$

$$= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2) \cap \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(3).$$

The following example show that the intersection of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A.

Example 6.1.3 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \emptyset & \text{if } x = 2, \\ \{0\} & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0, 0.5\} & \text{if } x = 0, \\ \{0, 0.5\} & \text{if } x = 1, \\ \{0, 5\} & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0, 0.5\} & \text{if } x = 0, \\ \{0, 0.5\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{0\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not a hesitant fuzzy UP-filter of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(3) &= \{0\} \\ &\not\supseteq \{0.5\} \\ &= \{0, 0.5\} \cap \{0.5\} \\ &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(0) \cap \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2) \\ &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2 \cdot 3) \cap \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2). \end{aligned}$$

The following example show that the intersection of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A.

Example 6.1.4 From a UP-algebra $A = \{0,1,2,3,4\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0,0.1\} & \text{if } x = 0, \\ \{0,0.1\} & \text{if } x = 1, \\ \{0.1\} & \text{if } x = 2, \\ \{0.1\} & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0,0.1\} & \text{if } x = 0, \\ \{0.1\} & \text{if } x = 1, \\ \{0,0,1\} & \text{if } x = 2, \\ \{0.1\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0,0.1\} & \text{if } x = 0, \\ \{0,0.1\} & \text{if } x = 1, \\ \{0,0.1\} & \text{if } x = 2, \\ \{0.1\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not a hesitant fuzzy UP-ideal of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(0\cdot 3) &= \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(3) \\ &= \{0.1\} \\ &\not\supseteq \{0,0.1\} \\ &= \{0,0.1\} \cap \{0,0.1\} \\ &= \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(2) \cap \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(2) \\ &= \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(0\cdot(1\cdot 3)) \cap \mathrm{h}_{\mathrm{H}_1\cap_T\mathrm{H}_2}(1). \end{aligned}$$

Theorem 6.1.5 *The intersection of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideals of A.*

Proof. Assume that H and G are hesitant fuzzy strongly UP-ideals of A. By Theorem 4.1.5, we have H and G are constant hesitant fuzzy sets of A. Thus $\mathrm{H} \cap_T \mathrm{G}$ is a constant hesitant fuzzy set of A. By Theorem 4.1.5, we can conclude that $\mathrm{H} \cap_T \mathrm{G}$ is a hesitant fuzzy strongly UP-ideal of A. □

The following example show that the intersection of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A.

Example 6.1.6 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.7\} & \text{if } x = 0, \\ [0.7, 0.8) & \text{if } x = 1, \\ [0.7, 0.9] & \text{if } x = 2, \\ [0.6, 0.9] & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.7\} & \text{if } x = 0, \\ \{0.7, 0.8\} & \text{if } x = 1, \\ \{0.7, 0.8\} & \text{if } x = 2, \\ \{0.7\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0.7\} & \text{if } x = 0, \\ [0.7, 0.8) & \text{if } x = 1, \\ [0.7, 0.9] & \text{if } x = 2, \\ [0.6, 0.7] & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(1 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2) \\ &= [0.7, 0.9] \\ &\nsubseteq [0.6, 0.8) \\ &= [0.7, 0.8) \cup [0.6, 0.7] \end{aligned}$$

$$= h_{H_1 \cap_T H_2}(1) \cup h_{H_1 \cap_T H_2}(3).$$

The following example show that the intersection of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A.

Example 6.1.7 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \{0\} & \text{if } x = 2, \\ \{0\} & \text{if } x = 3 \end{cases} \text{ and } h_{H_2}(x) = \begin{cases} \{0.2\} & \text{if } x = 0, \\ \{0, 0.2\} & \text{if } x = 1, \\ \{0.2, 0.8, 1\} & \text{if } x = 2, \\ \{0.2, 1\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cap_T H_2(x) = \begin{cases} \{0.2\} & \text{if } x = 0, \\ \{0, 0.2\} & \text{if } x = 1, \\ \{0\} & \text{if } x = 2, \\ \{0\} & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap_T H_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$h_{H_1 \cap_T H_2}(0) = \{0.2\} \nsubseteq \{0\} = h_{H_1 \cap_T H_2}(2).$$

The following example show that the intersection of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A.

Example 6.1.8 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define

two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{1\} & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \{0.8, 1\} & \text{if } x = 2, \\ \{0.8, 1\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \emptyset & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{1\} & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \{0.8, 1\} & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$\mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(0) = \{1\} \nsubseteq \{0.5\} = \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(1).$$

Theorem 6.1.9 *The intersection of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideals of A.*

Proof. Assume that H and G are anti-hesitant fuzzy strongly UP-ideals of A. By Theorem 4.2.9, we have H and G are constant hesitant fuzzy sets of A. Thus $\mathrm{H} \cap_T \mathrm{G}$ is a constant hesitant fuzzy set of A. By Theorem 4.2.9, we can conclude that $\mathrm{H} \cap_T \mathrm{G}$ is an anti-hesitant fuzzy strongly UP-ideal of A. □

The following example show that the union of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A.

Example 6.1.10 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define

two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0, 0.5, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{0\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0, 0.5\} & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \{0, 5\} & \text{if } x = 2, \\ \{0, 0.5\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cup_T \mathrm{H}_2(x) = \begin{cases} \{0, 0.5, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{0, 0.5\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup_T \mathrm{H}_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(2 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(1) \\ &= \{1\} \\ &\not\supseteq \{0.5\} \\ &= \{0.5\} \cap \{0, 0.5\} \\ &= \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(2) \cap \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(3). \end{aligned}$$

The following example show that the union of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A.

Example 6.1.11 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0,1\} & \text{if } x = 0, \\ \{0\} & \text{if } x = 1, \\ \emptyset & \text{if } x = 2, \\ \{1\} & \text{if } x = 3 \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0.5,1\} & \text{if } x = 0, \\ \{0.5,1\} & \text{if } x = 1, \\ \{0,5\} & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$H_1 \cup_T H_2(x) = \begin{cases} \{0.5,1\} & \text{if } x = 0, \\ \{0.5,1\} & \text{if } x = 1, \\ \{0.5\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cup_T H_2$ is not a hesitant fuzzy UP-filter of A because

$$\begin{aligned} h_{H_1 \cup_T H_2}(3) &= \{1\} \\ &\not\supseteq \{0.5\} \\ &= \{0.5,1\} \cap \{0.5\} \\ &= h_{H_1 \cup_T H_2}(0) \cap h_{H_1 \cup_T H_2}(2) \\ &= h_{H_1 \cup_T H_2}(2 \cdot 3) \cap h_{H_1 \cup_T H_2}(2). \end{aligned}$$

The following example show that the union of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A.

Example 6.1.12 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define

two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{0.9, 1\} & \text{if } x = 1, \\ \{0.9\} & \text{if } x = 2, \\ \{0.9\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{0.9\} & \text{if } x = 1, \\ \{0, 9, 1\} & \text{if } x = 2, \\ \{0.9\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{0.9, 1\} & \text{if } x = 1, \\ \{0.9, 1\} & \text{if } x = 2, \\ \{0.9\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap_T \mathrm{H}_2$ is not a hesitant fuzzy UP-ideal of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(0 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(3) \\ &= \{0.9\} \\ &\not\supseteq \{0.9, 1\} \\ &= \{0.9, 1\} \cap \{0.9, 1\} \\ &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2) \cap \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(2) \\ &= \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(0 \cdot (1 \cdot 3)) \cap \mathrm{h}_{\mathrm{H}_1 \cap_T \mathrm{H}_2}(1). \end{aligned}$$

Theorem 6.1.13 *The union of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideals of A.*

Proof. Assume that H and G are hesitant fuzzy strongly UP-ideals of A. By Theorem 4.1.5, we have H and G are constant hesitant fuzzy sets of A. Thus $\mathrm{H} \cup_T \mathrm{G}$ is a constant hesitant fuzzy set of A. By Theorem 4.1.5, we can conclude that $\mathrm{H} \cup_T \mathrm{G}$ is a hesitant fuzzy strongly UP-ideal of A. $\square$

The following example show that the union of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A.

Example 6.1.14 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ (0.2, 0.3] & \text{if } x = 1, \\ [0.1, 0.3] & \text{if } x = 2, \\ [0.1, 0.4] & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ \{0.2, 0.3\} & \text{if } x = 1, \\ \{0.2, 0.3\} & \text{if } x = 2, \\ \{0.3\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cup_T \mathrm{H}_2(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ (0.2, 0.3] & \text{if } x = 1, \\ [0.1, 0.3] & \text{if } x = 2, \\ [0.3, 0.4] & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup_T \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(1 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(2) \\ &= [0.1, 0.3] \\ &\nsubseteq (0.2, 0.4] \\ &= (0.2, 0.3] \cup [0.3, 0.4] \\ &= \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(1) \cup \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(3). \end{aligned}$$

The following example show that the union of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A.

Example 6.1.15 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \{0\} & \text{if } x = 1, \\ \{1\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.8\} & \text{if } x = 0, \\ \{0.8, 1\} & \text{if } x = 1, \\ \{0, 0.2, 0.8\} & \text{if } x = 2, \\ \{0, 0.8\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cap_T \mathrm{H}_2(x) = \begin{cases} \{0.8\} & \text{if } x = 0, \\ \{0.8, 1\} & \text{if } x = 1, \\ \{1\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup_T \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$\mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(0) = \{0.8\} \nsubseteq \{1\} = \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(2).$$

The following example show that the union of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A.

Example 6.1.16 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{0\} & \text{if } x = 1, \\ \{0, 0.2\} & \text{if } x = 2, \\ \{0, 0.2\} & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \emptyset & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A. We thus obtain the intersection of H_1 and H_2 as follows:

$$\mathrm{H}_1 \cup_T \mathrm{H}_2(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{0.5\} & \text{if } x = 1, \\ \{0, 0.2\} & \text{if } x = 2, \\ \{0.5\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup_T \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$\mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(0) = \{1\} \nsubseteq \{0.5\} = \mathrm{h}_{\mathrm{H}_1 \cup_T \mathrm{H}_2}(1).$$

Theorem 6.1.17 *The union of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideals of A.*

Proof. Assume that H and G are anti-hesitant fuzzy strongly UP-ideals of A. By Theorem 4.2.9, we have H and G are constant hesitant fuzzy sets of A. Thus $\mathrm{H} \cup_T \mathrm{G}$ is a constant hesitant fuzzy set of A. By Theorem 4.2.9, we can conclude that $\mathrm{H} \cup_T \mathrm{G}$ is an anti-hesitant fuzzy strongly UP-ideal of A. □

We see that the intersection and the union can not be applied to several types of hesitant fuzzy sets on UP-algebras except hesitant fuzzy strongly UP-ideals and anti-hesitant fuzzy strongly UP-ideals. That make we will consider the others.

The concepts of the hesitant union, and the hesitant intersection of two hesitant fuzzy sets were introduced by Jun [15] in 2015. We will introduce the notions of the hesitant union and the hesitant intersection of any hesitant fuzzy sets and apply to UP-algebras.

Definition 6.1.18 Let $\{\mathrm{H}_i \mid i \in I\}$ be a family of hesitant fuzzy sets on a reference set X. We define the hesitant intersection operation $\cap$ on $\{\mathrm{H}_i \mid i \in I\}$ by

$$\bigcap_{i\in I} \mathrm{H}_i = \{(x, \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x)) \mid x \in X\},$$

which is called the *hesitant intersection* of hesitant fuzzy sets. In particular, $\mathrm{H}_i \cap \mathrm{H}_k = \{(x, \mathrm{h}_{\mathrm{H}_i}(x) \cap \mathrm{h}_{\mathrm{H}_k}(x)) \mid x \in X\}$ for all $i, k \in I$.

Theorem 6.1.19 *The hesitant intersection of hesitant fuzzy UP-subalgebras of A is a hesitant fuzzy UP-subalgebra of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of hesitant fuzzy UP-subalgebras of A. Let $x, y \in A$. Then $\mathrm{h}_{\mathrm{H}_i}(x \cdot y) \supseteq \mathrm{h}_{\mathrm{H}_i}(x) \cap \mathrm{h}_{\mathrm{H}_i}(y)$ for all $i \in I$. Thus

$$\begin{aligned}
\mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(x \cdot y) &= \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x \cdot y) \\
&\supseteq \bigcap_{i\in I} (\mathrm{h}_{\mathrm{H}_i}(x) \cap \mathrm{h}_{\mathrm{H}_i}(y)) \\
&= \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x) \cap \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(y) \\
&= \mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(x) \cap \mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(y).
\end{aligned}$$

Hence, $\bigcap_{i\in I} \mathrm{H}_i$ is a hesitant fuzzy UP-subalgebra of A. $\square$

Theorem 6.1.20 *The hesitant intersection of hesitant fuzzy UP-filters of A is a hesitant fuzzy UP-filter of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of hesitant fuzzy UP-filters of A. Let $x, y \in A$. Then $\mathrm{h}_{\mathrm{H}_i}(0) \supseteq \mathrm{h}_{\mathrm{H}_i}(x)$ and $\mathrm{h}_{\mathrm{H}_i}(y) \supseteq \mathrm{h}_{\mathrm{H}_i}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}_i}(x)$ for all $i \in I$. Thus $\mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(0) = \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(0) \supseteq \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x) = \mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(x)$ and

$$\mathrm{h}_{\bigcap_{i\in I} \mathrm{H}_i}(y) = \bigcap_{i\in I} \mathrm{h}_{\mathrm{H}_i}(y)$$

$$\begin{aligned}
&\supseteq \bigcap_{i\in I}(\mathrm{h}_{\mathrm{H}_i}(x\cdot y)\cap \mathrm{h}_{\mathrm{H}_i}(x))\\
&= \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(x\cdot y)\cap \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(x)\\
&= \mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(x\cdot y)\cap \mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(x).
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{H}_i$ is a hesitant fuzzy UP-filter of A. □

Theorem 6.1.21 *The hesitant intersection of hesitant fuzzy UP-ideals of A is a hesitant fuzzy UP-ideal of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i\in I\}$ is a family of hesitant fuzzy UP-ideals of A. Let $x,y\in A$. Then $\mathrm{h}_{\mathrm{H}_i}(0)\supseteq \mathrm{h}_{\mathrm{H}_i}(x)$ and $\mathrm{h}_{\mathrm{H}_i}(x\cdot z)\supseteq \mathrm{h}_{\mathrm{H}_i}(x\cdot(y\cdot z))\cap \mathrm{h}_{\mathrm{H}_i}(y)$ for all $i\in I$. Thus $\mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(0)=\bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(0)\supseteq \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(x)=\mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(x)$ and

$$\begin{aligned}
\mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(x\cdot z) &= \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(x\cdot z)\\
&\supseteq \bigcap_{i\in I}(\mathrm{h}_{\mathrm{H}_i}(x\cdot(y\cdot z))\cap \mathrm{h}_{\mathrm{H}_i}(y))\\
&= \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(x\cdot(y\cdot z))\cap \bigcap_{i\in I}\mathrm{h}_{\mathrm{H}_i}(y)\\
&= \mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(x\cdot(y\cdot z))\cap \mathrm{h}_{\bigcap_{i\in I}\mathrm{H}_i}(y).
\end{aligned}$$

Hence, $\bigcap_{i\in I}\mathrm{H}_i$ is a hesitant fuzzy UP-ideal of A. □

Theorem 6.1.22 *The hesitant intersection of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideal of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i\in I\}$ is a family of hesitant fuzzy strongly UP-ideals of A. Then H_i is a hesitant fuzzy strongly UP-ideal of A for all $i\in I$. By Theorem 4.1.5, we have H_i is a constant hesitant fuzzy set on A for all $i\in I$. Thus $\bigcap_{i\in I}\mathrm{H}_i$

is a constant hesitant fuzzy set on A. By Theorem 4.1.5, we know that $\bigcap_{i\in I} \mathrm{H}_i$ is a hesitant fuzzy strongly UP-ideal of A. $\square$

The following example show that the hesitant intersection of anti-hesitant fuzzy UP-subalgebras of A is not an anti-hesitant fuzzy UP-subalgebra of A.

Example 6.1.23 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ [0.3, 0.4) & \text{if } x = 1, \\ [0.3, 0.5] & \text{if } x = 2, \\ [0, 0.5] & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ \{0.3, 0.4\} & \text{if } x = 1, \\ \{0.3, 0.4\} & \text{if } x = 2, \\ \{0.3\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-subalgebras of A. We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$\mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(x) = \begin{cases} \{0.3\} & \text{if } x = 0, \\ \{0.3\} & \text{if } x = 1, \\ \{0.3, 0.4\} & \text{if } x = 2, \\ \{0.3\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(1 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(2) \\ &= \{0.3, 0.4\} \\ &\nsubseteq \{0.3\} \\ &= \{0.3\} \cup \{0.3\} \\ &= \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(1) \cup \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(3). \end{aligned}$$

The following example show that the hesitant intersection of anti-hesitant fuzzy UP-filters of A is not an anti-hesitant fuzzy UP-filter of A.

Example 6.1.24 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.6\} & \text{if } x = 0, \\ \{0.6, 0.7\} & \text{if } x = 1, \\ \{0.5, 0.6\} & \text{if } x = 2, \\ \{0.5, 0.6\} & \text{if } x = 3 \end{cases} \text{ and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.6\} & \text{if } x = 0, \\ \{0.5, 0.6, 0.7\} & \text{if } x = 1, \\ \{0.4, 0.5, 0.6\} & \text{if } x = 2, \\ \{0.4, 0.6\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-filters of A. We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$\mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(x) = \begin{cases} \{0.6\} & \text{if } x = 0, \\ \{0.6, 0.7\} & \text{if } x = 1, \\ \{0.5, 0.6\} & \text{if } x = 2, \\ \{0.6\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cap \mathrm{H}_2$ is not an anti-hesitant fuzzy UP-filter of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(2) &= \{0.5, 0.6\} \\ &\not\subseteq \{0.6, 0.7\} \\ &= \{0.6\} \cup \{0.6, 0.7\} \\ &= \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(3) \cup \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(1) \\ &= \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(1 \cdot 2) \cup \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(1). \end{aligned}$$

The following example show that the hesitant intersection of anti-hesitant fuzzy UP-ideals of A is not an anti-hesitant fuzzy UP-ideal of A.

Example 6.1.25 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define

two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$h_{H_1}(x) = \begin{cases} \{0.9\} & \text{if } x = 0, \\ \{0.9\} & \text{if } x = 1, \\ \{0.9, 1\} & \text{if } x = 2, \\ \{0.9, 1\} & \text{if } x = 3 \end{cases} \quad \text{and } h_{H_2}(x) = \begin{cases} \{0.9\} & \text{if } x = 0, \\ \{0.9, 1\} & \text{if } x = 1, \\ \{0.9\} & \text{if } x = 2, \\ \{0.9, 1\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are anti-hesitant fuzzy UP-ideals of A. We thus obtain the hesitant intersection of H_1 and H_2 as follows:

$$h_{H_1 \cap H_2}(x) = \begin{cases} \{0.9\} & \text{if } x = 0, \\ \{0.9\} & \text{if } x = 1, \\ \{0.9\} & \text{if } x = 2, \\ \{0.9, 1\} & \text{if } x = 3. \end{cases}$$

Therefore, $H_1 \cap H_2$ is not an anti-hesitant fuzzy UP-ideal of A because

$$\begin{aligned} h_{H_1 \cap H_2}(0 \cdot 3) &= h_{H_1 \cap H_2}(3) \\ &= \{0.9, 1\} \\ &\not\subseteq \{0.9\} \\ &= \{0.9\} \cup \{0.9\} \\ &= h_{H_1 \cap H_2}(1) \cup h_{H_1 \cap H_2}(2) \\ &= h_{H_1 \cap H_2}(0 \cdot (2 \cdot 3)) \cup h_{H_1 \cap H_2}(2). \end{aligned}$$

Theorem 6.1.26 *The hesitant intersection of anti-hesitant fuzzy strongly UP-ideals of A is an anti-hesitant fuzzy strongly UP-ideal of A.*

Proof. Assume that $\{H_i \mid i \in I\}$ is a family of anti-hesitant fuzzy strongly UP-ideals of A. Then H_i is an anti-hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 4.2.9, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$.

Thus $\bigcap_{i\in I} \mathrm{H}_i$ is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we know that $\bigcap_{i\in I} \mathrm{H}_i$ is an anti-hesitant fuzzy strongly UP-ideal of A. □

Definition 6.1.27 Let $\{\mathrm{H}_i \mid i \in I\}$ be a family of hesitant fuzzy sets on a reference set X. We define the hesitant union operation $\cup$ on $\{\mathrm{H}_i \mid i \in I\}$ by

$$\bigcup_{i\in I} \mathrm{H}_i = \{(x, \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x)) \mid x \in X\},$$

which is called the *hesitant union* of hesitant fuzzy sets. In particular, $\mathrm{H}_i \cup \mathrm{H}_k = \{(x, \mathrm{h}_{\mathrm{H}_i}(x) \cup \mathrm{h}_{\mathrm{H}_k}(x)) \mid x \in X\}$ for all $i, k \in I$.

The following example show that the hesitant union of hesitant fuzzy UP-subalgebras of A is not a hesitant fuzzy UP-subalgebra of A.

Example 6.1.28 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} [0.6, 0.9] & \text{if } x = 0, \\ \{0.8\} & \text{if } x = 1, \\ \{0.7\} & \text{if } x = 2, \\ \{0.6\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} [0.6, 0.8] & \text{if } x = 0, \\ \{0.7\} & \text{if } x = 1, \\ \{0.7\} & \text{if } x = 2, \\ [0.6, 0.8] & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-subalgebras of A. We thus obtain the hesitant union of H_1 and H_2 as follows:

$$\mathrm{h}_{\mathrm{H}_1\cup\mathrm{H}_2}(x) = \begin{cases} [0.6, 0.9] & \text{if } x = 0, \\ \{0.7, 0.8\} & \text{if } x = 1, \\ \{0.7\} & \text{if } x = 2, \\ [0.6, 0.8] & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup \mathrm{H}_2$ is not a hesitant fuzzy UP-subalgebra of A because

$$\begin{aligned} \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cap \mathrm{H}_2}(2) \\ &= \{0.7\} \\ &\not\supseteq \{0.7, 0.8\} \\ &= \{0.7, 0.8\} \cap [0.6, 0.8] \\ &= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1) \cap \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(3). \end{aligned}$$

The following example show that the hesitant union of hesitant fuzzy UP-filters of A is not a hesitant fuzzy UP-filter of A.

Example 6.1.29 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 5.1.19. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0\} \cup [0.8, 1] & \text{if } x = 0, \\ \{0\} \cup [0.8, 1] & \text{if } x = 1, \\ \{0.8\} & \text{if } x = 2, \\ \{0.8\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} [0, 0.8] & \text{if } x = 0, \\ \{0.8\} & \text{if } x = 1, \\ \emptyset & \text{if } x = 2, \\ \{0\} & \text{if } x = 3. \end{cases}$$

Then H_1 and H_2 are hesitant fuzzy UP-filters of A. We thus obtain the hesitant union of H_1 and H_2 as follows:

$$\mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(x) = \begin{cases} [0, 1] & \text{if } x = 0, \\ \{0\} \cup [0.8, 1] & \text{if } x = 1, \\ \{0.8\} & \text{if } x = 2, \\ \{0, 0.8\} & \text{if } x = 3. \end{cases}$$

Therefore, $\mathrm{H}_1 \cup \mathrm{H}_2$ is not a hesitant fuzzy UP-filter of A because

$$\mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(2) = \{0.8\}$$

$$
\begin{aligned}
&\not\supseteq \{0, 0.8\} \\
&= \{0, 0.8\} \cap (\{0\} \cup [0.8, 1]) \\
&= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(3) \cap \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1) \\
&= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1 \cdot 2) \cap \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1).
\end{aligned}
$$

The following example show that the hesitant union of hesitant fuzzy UP-ideals of A is not a hesitant fuzzy UP-ideal of A.

Example 6.1.30 From a UP-algebra $A = \{0, 1, 2, 3\}$ of Example 6.1.2. We define two hesitant fuzzy sets H_1 and H_2 on A as follows:

$$
\mathrm{h}_{\mathrm{H}_1}(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{0.9, 1\} & \text{if } x = 1, \\ \{1\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3 \end{cases} \quad \text{and } \mathrm{h}_{\mathrm{H}_2}(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{1\} & \text{if } x = 1, \\ \{0.9, 1\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3. \end{cases}
$$

Then H_1 and H_2 are hesitant fuzzy UP-ideals of A. We thus obtain the hesitant union of H_1 and H_2 as follows:

$$
\mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(x) = \begin{cases} \{0.9, 1\} & \text{if } x = 0, \\ \{0.9, 1\} & \text{if } x = 1, \\ \{0.9, 1\} & \text{if } x = 2, \\ \{1\} & \text{if } x = 3. \end{cases}
$$

Therefore, $\mathrm{H}_1 \cup \mathrm{H}_2$ is not a hesitant fuzzy UP-ideal of A because

$$
\begin{aligned}
\mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(0 \cdot 3) &= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(3) \\
&= \{1\} \\
&\not\supseteq \{0.9, 1\}
\end{aligned}
$$

$$= \{0.9, 1\} \cap \{0.9, 1\}$$
$$= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(1) \cap \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(2)$$
$$= \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(0 \cdot (2 \cdot 3)) \cap \mathrm{h}_{\mathrm{H}_1 \cup \mathrm{H}_2}(2).$$

Theorem 6.1.31 *The hesitant union of hesitant fuzzy strongly UP-ideals of A is a hesitant fuzzy strongly UP-ideal of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of hesitant fuzzy strongly UP-ideals of A. Then H_i is a hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 4.1.5, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcup_{i \in I} \mathrm{H}_i$ is a constant hesitant fuzzy set on A. By Theorem 4.1.5, we know that $\bigcup_{i \in I} \mathrm{H}_i$ is a hesitant fuzzy strongly UP-ideal of A. □

Theorem 6.1.32 *The hesitant union of anti-hesitant fuzzy UP-subalgebras of A is an anti-hesitant fuzzy UP-subalgebra of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-subalgebras of A. Let $x, y \in A$. Then $\mathrm{h}_{\mathrm{H}_i}(x \cdot y) \subseteq \mathrm{h}_{\mathrm{H}_i}(x) \cup \mathrm{h}_{\mathrm{H}_i}(y)$ for all $i \in I$. Thus

$$\begin{aligned}
\mathrm{h}_{\bigcup_{i \in I} \mathrm{H}_i}(x \cdot y) &= \bigcup_{i \in I} \mathrm{h}_{\mathrm{H}_i}(x \cdot y) \\
&\subseteq \bigcup_{i \in I} (\mathrm{h}_{\mathrm{H}_i}(x) \cup \mathrm{h}_{\mathrm{H}_i}(y)) \\
&= \bigcup_{i \in I} \mathrm{h}_{\mathrm{H}_i}(x) \cup \bigcup_{i \in I} \mathrm{h}_{\mathrm{H}_i}(y) \\
&= \mathrm{h}_{\bigcup_{i \in I} \mathrm{H}_i}(x) \cup \mathrm{h}_{\bigcup_{i \in I} \mathrm{H}_i}(y).
\end{aligned}$$

Hence, $\bigcup_{i \in I} \mathrm{H}_i$ is an anti-hesitant fuzzy UP-subalgebra of A. □

Theorem 6.1.33 *The hesitant union of anti-hesitant fuzzy UP-filters of A is an anti-hesitant fuzzy UP-filter of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-filters of A. Let $x, y \in A$. Then $\mathrm{h}_{\mathrm{H}_i}(0) \subseteq \mathrm{h}_{\mathrm{H}_i}(x)$ and $\mathrm{h}_{\mathrm{H}_i}(y) \subseteq \mathrm{h}_{\mathrm{H}_i}(x \cdot y) \cup \mathrm{h}_{\mathrm{H}_i}(x)$ for all $i \in I$. Thus $\mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(0) = \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(0) \subseteq \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x) = \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x)$ and

$$\begin{aligned}
\mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(y) &= \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(y) \\
&\subseteq \bigcup_{i\in I} (\mathrm{h}_{\mathrm{H}_i}(x \cdot y) \cup \mathrm{h}_{\mathrm{H}_i}(x)) \\
&= \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x \cdot y) \cup \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x) \\
&= \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x \cdot y) \cup \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x).
\end{aligned}$$

Hence, $\bigcup_{i\in I} \mathrm{H}_i$ is an anti-hesitant fuzzy UP-filter of A. □

Theorem 6.1.34 *The hesitant union of anti-hesitant fuzzy UP-ideals of A is an anti-hesitant fuzzy UP-ideal of A.*

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of anti-hesitant fuzzy UP-ideals of A. Let $x, y \in A$. Then $\mathrm{h}_{\mathrm{H}_i}(0) \subseteq \mathrm{h}_{\mathrm{H}_i}(x)$ and $\mathrm{h}_{\mathrm{H}_i}(x \cdot z) \subseteq \mathrm{h}_{\mathrm{H}_i}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\mathrm{H}_i}(y)$ for all $i \in I$. Thus $\mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(0) = \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(0) \subseteq \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x) = \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x)$ and

$$\begin{aligned}
\mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x \cdot z) &= \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x \cdot z) \\
&\subseteq \bigcup_{i\in I} (\mathrm{h}_{\mathrm{H}_i}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\mathrm{H}_i}(y)) \\
&= \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(x \cdot (y \cdot z)) \cup \bigcup_{i\in I} \mathrm{h}_{\mathrm{H}_i}(y) \\
&= \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\bigcup_{i\in I}\mathrm{H}_i}(y).
\end{aligned}$$

Hence, $\bigcup_{i\in I} \mathrm{H}_i$ is an anti-hesitant fuzzy UP-ideal of A. □

Theorem 6.1.35 *The hesitant union of anti-hesitant fuzzy strongly UP-ideals of*

A is an anti-hesitant fuzzy strongly UP-ideal of A.

Proof. Assume that $\{\mathrm{H}_i \mid i \in I\}$ is a family of anti-hesitant fuzzy strongly UP-ideals of A. Then H_i is an anti-hesitant fuzzy strongly UP-ideal of A for all $i \in I$. By Theorem 4.2.9, we have H_i is a constant hesitant fuzzy set on A for all $i \in I$. Thus $\bigcup_{i\in I} \mathrm{H}_i$ is a constant hesitant fuzzy set on A. By Theorem 4.2.9, we know that $\bigcup_{i\in I} \mathrm{H}_i$ is an anti-hesitant fuzzy strongly UP-ideal of A. □

6.2 Operations on hesitant fuzzy soft sets

The concepts of the "AND" operation and the "OR" operation on two hesitant fuzzy soft sets and the union and the intersection of two hesitant fuzzy soft sets were introduced by Wang et al. [41] in 2014.

Definition 6.2.1 Let $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ be two hesitant fuzzy soft sets over a reference set X. The AND *operation* on $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = Y_1 \times Y_2$, and

(2) $\widetilde{\mathrm{H}}[p_1, p_2] = \widetilde{\mathrm{H}}_1[p_1] \cap_T \widetilde{\mathrm{H}}_2[p_2]$ for any $(p_1, p_2) \in Y$.

We write $(\widetilde{\mathrm{H}}_1, Y_1)\widetilde{\wedge}(\widetilde{\mathrm{H}}_2, Y_2) = (\widetilde{\mathrm{H}}, Y)$.

Definition 6.2.2 Let $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ be two hesitant fuzzy soft sets over a reference set X. The OR *operation* on $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = Y_1 \times Y_2$, and

(2) $\widetilde{\mathrm{H}}[p_1, p_2] = \widetilde{\mathrm{H}}_1[p_1] \cup_T \widetilde{\mathrm{H}}_2[p_2]$ for any $(p_1, p_2) \in Y$.

We write $(\widetilde{\mathrm{H}}_1, Y_1)\widetilde{\vee}(\widetilde{\mathrm{H}}_2, Y_2) = (\widetilde{\mathrm{H}}, Y)$.

Definition 6.2.3 Let $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ be two hesitant fuzzy soft sets over a reference set X. The *union* of $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = Y_1 \cup Y_2$, and

(2) for any $p \in Y$,

$$\widetilde{\mathrm{H}}[p] = \begin{cases} \widetilde{\mathrm{H}}_1[p] & \text{if } p \in Y_1 \backslash Y_2, \\ \widetilde{\mathrm{H}}_2[p] & \text{if } p \in Y_2 \backslash Y_1, \\ \widetilde{\mathrm{H}}_1[p] \cup_T \widetilde{\mathrm{H}}_2[p] & \text{if } p \in Y_1 \cap Y_2. \end{cases}$$

We write $(\widetilde{\mathrm{H}}_1, Y_1)\widetilde{\cup}(\widetilde{\mathrm{H}}_2, Y_2) = (\widetilde{\mathrm{H}}, Y)$.

Definition 6.2.4 Let $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ be two hesitant fuzzy soft sets over a reference set X. The *intersection* of $(\widetilde{\mathrm{H}}_1, Y_1)$ and $(\widetilde{\mathrm{H}}_2, Y_2)$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = Y_1 \cap Y_2 \neq \emptyset$, and

(2) $\widetilde{\mathrm{H}}[p] = \widetilde{\mathrm{H}}_1[p] \cap_T \widetilde{\mathrm{H}}_2[p]$ for all $p \in Y$.

We write $(\widetilde{\mathrm{H}}_1, Y_1)\widetilde{\cap}(\widetilde{\mathrm{H}}_2, Y_2) = (\widetilde{\mathrm{H}}, Y)$.

We will introduce the notions of the union, the intersection, the restricted union, the extended intersection, the AND, and the OR of any hesitant fuzzy soft sets and apply to UP-algebras.

Definition 6.2.5 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The *restricted union* of $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \bigcap_{i \in I} Y_i \neq \emptyset$, and

(2) $\widetilde{\mathrm{H}}[p] = \bigcup_{i \in I} \widetilde{\mathrm{H}}_i[p]$ for all $p \in Y$.

We write $\widetilde{\bigcup_{i \in I}} r(\widetilde{\mathrm{H}}_i, Y_i) = (\widetilde{\mathrm{H}}, Y)$.

Theorem 6.2.6 *The restricted union of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is an anti-hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is the restricted union of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$ and let $p \in Y$. Then $p \in Y = \bigcap_{i \in I} Y_i$ and $\widetilde{\mathrm{H}}[p] = \bigcup_{i \in I} \widetilde{\mathrm{H}}_i[p]$. Thus $p \in Y_i$ for all $i \in I$ and so $\widetilde{\mathrm{H}}_i[p]$ is a hesitant fuzzy UP-subalgebras of A for all $i \in I$. By Theorem 6.1.32, we have $\widetilde{\mathrm{H}}[p] = \bigcup_{i \in I} \widetilde{\mathrm{H}}_i[p]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.6, we can use Theorem 6.1.33, Theorem 6.1.34, Theorem 6.1.35 and Theorem 6.1.31, respectively, to prove the following theorem.

Theorem 6.2.7 *The restricted union of anti-hesitant fuzzy soft UP-filters (resp., anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-filter (resp., an anti-hesitant fuzzy soft UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal, a hesitant fuzzy soft strongly UP-ideal) of A.*

By Example 6.1.28 (resp., Example 6.1.29, Example 6.1.30), we can imply that the restricted union of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 6.2.8 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The *union* $\widetilde{\bigcup_{i\in I}}(\widetilde{\mathrm{H}}_i, Y_i)$ of hesitant fuzzy soft sets is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \bigcup_{i\in I} Y_i$, and

(2) for any $p \in Y$ with $p \in \bigcap_{j\in J} Y_j \backslash \bigcup_{k\in I\backslash J} Y_k$ where $\emptyset \neq J \subseteq I$, $\widetilde{\mathrm{H}}[p] = \bigcup_{j\in J} \widetilde{\mathrm{H}}_j[p]$.

Theorem 6.2.9 *The union of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is an anti-hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is the union of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$. Then $Y = \bigcup_{i\in I} Y_i$. Let $p \in Y$.

If $p \in \bigcap_{i\in I} Y_i \neq \emptyset$, then it follows from Theorem 6.2.6 that $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A.

If $p \in Y_j \backslash \bigcup_{k\neq j} Y_k = Y_j \backslash \bigcup_{k\in I\backslash\{j\}} Y_k$ where $j \in I$, then $\widetilde{\mathrm{H}}[p] = \bigcup_{j\in\{j\}} \widetilde{\mathrm{H}}_j[p] = \widetilde{\mathrm{H}}_j[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A.

If $p \in \bigcap_{j\in J} Y_j \backslash \bigcup_{k\in I\backslash J} Y_k$ where $\emptyset \neq J \subseteq I$, then $\widetilde{\mathrm{H}}[p] = \bigcup_{j\in J} \widetilde{\mathrm{H}}_j[p]$ and $p \in Y_j$ for all $j \in J$ but $p \notin Y_k$ for some $k \in I\backslash J$. Thus $\widetilde{\mathrm{H}}_j[p]$ is an anti-hesitant fuzzy UP-subalgebra of A for all $j \in J$. By Theorem 6.1.32, we have $\widetilde{\mathrm{H}}[p] = \bigcup_{j\in J} \widetilde{\mathrm{H}}_j[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A.

Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.9, we can use Theorem 6.2.7 and Theorem 6.1.33 (resp., Theorem 6.1.34, Theorem 6.1.35, Theorem 6.1.31) to prove that the union of anti-hesitant fuzzy soft UP-filters (resp., anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-filter (resp., an anti-hesitant fuzzy soft UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal, a hesitant fuzzy soft strongly UP-ideal) of A. By the way, we also confirm that the union of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 6.2.10 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The *intersection* of $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \bigcap_{i \in I} Y_i \neq \emptyset$, and

(2) $\widetilde{\mathrm{H}}[p] = \bigcap_{i \in I} \widetilde{\mathrm{H}}_i[p]$ for all $p \in Y$.

We write $\widetilde{\bigcap_{i \in I}}(\widetilde{\mathrm{H}}_i, Y_i) = (\widetilde{\mathrm{H}}, Y)$

Theorem 6.2.11 *The intersection of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$.

Assume that $(\widetilde{\mathrm{H}}, Y)$ is the intersection of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$ and let $p \in Y$. Then $p \in Y = \bigcap_{i\in I} Y_i$ and $\widetilde{\mathrm{H}}[p] = \bigcap_{i\in I} \widetilde{\mathrm{H}}_i[p]$. Thus $p \in Y_i$ for all $i \in I$ and so $\widetilde{\mathrm{H}}_i[p]$ is a hesitant fuzzy UP-subalgebras of A for all $i \in I$. By Theorem 6.1.19, we have $\widetilde{\mathrm{H}}[p] = \bigcap_{i\in I} \widetilde{\mathrm{H}}_i[p]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.11, we can use Theorem 6.1.20, Theorem 6.1.21, Theorem 6.1.22 and Theorem 6.1.26, respectively, to prove the following theorem.

Theorem 6.2.12 *The intersection of hesitant fuzzy soft UP-filters (resp., hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-filter (resp., a hesitant fuzzy soft UP-ideals, a hesitant fuzzy soft strongly UP-ideals, an anti-hesitant fuzzy soft strongly UP-ideals) of A.*

By Example 6.1.23 (resp., Example 6.1.24, Example 6.1.25), we can imply that the intersection of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals) of A is not an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal) of A in general.

Definition 6.2.13 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The *extended intersection* of $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \bigcup_{i\in I} Y_i$, and

(2) for any $p \in Y$ with $p \in \bigcap_{j\in J} Y_j \setminus \bigcup_{k\in I\setminus J} Y_k$ where $\emptyset \neq J \subseteq I$, $\widetilde{\mathrm{H}}[p] = \bigcap_{j\in J} \widetilde{\mathrm{H}}_j[p]$.

We write $\widetilde{\bigcap_{i\in I}}e(\widetilde{\mathrm{H}}_i, Y_i) = (\widetilde{\mathrm{H}}, Y)$.

Theorem 6.2.14 *The extended intersection of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A for all $p_i \in Y_i$ and $i \in I$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is the extended intersection of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$. Then $Y = \bigcup_{i\in I} Y_i$. Let $p \in Y$.

If $p \in \bigcap_{i\in I} Y_i \neq \emptyset$, then it follows from Theorem 6.2.11 that $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A.

If $p \in Y_j \backslash \bigcup_{k\neq j} Y_k = Y_j \backslash \bigcup_{k\in I\backslash\{j\}} Y_k$ where $j \in I$, then $\widetilde{\mathrm{H}}[p] = \bigcap_{j\in\{j\}} \widetilde{\mathrm{H}}_j[p] = \widetilde{\mathrm{H}}_j[p]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A.

If $p \in \bigcap_{j\in J} Y_j \backslash \bigcup_{k\in I\backslash J} Y_k$ where $\emptyset \neq J \subseteq I$, then $\widetilde{\mathrm{H}}[p] = \bigcap_{j\in J} \widetilde{\mathrm{H}}_j[p]$ and $p \in Y_j$ for all $j \in J$ but $p \notin Y_k$ for some $k \in I\backslash J$. Thus $\widetilde{\mathrm{H}}_j[p]$ is a hesitant fuzzy UP-subalgebra of A for all $j \in J$. By Theorem 6.1.19, we have $\widetilde{\mathrm{H}}[p] = \bigcap_{j\in J} \widetilde{\mathrm{H}}_j[p]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra of A.

Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.14, we can use Theorem 6.2.12 and Theorem 6.1.20 (resp., 6.1.21, Theorem 6.1.22, Theorem 6.1.23) to prove that the extended intersection of hesitant fuzzy soft UP-filters (resp., hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly

UP-ideals) of A is a hesitant fuzzy soft UP-filter (resp., a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal) of A. By the way, we also confirm that the extended intersection of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

Definition 6.2.15 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The AND of $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \prod_{i \in I} Y_i$, and

(2) $\widetilde{\mathrm{H}}[(p_i)_{i \in I}] = \bigcap_{i \in I} \widetilde{\mathrm{H}}_i[p_i]$ for all $(p_i)_{i \in I} \in Y$.

We write $\widetilde{\bigwedge_{i \in I}}(\widetilde{\mathrm{H}}_i, Y_i) = (\widetilde{\mathrm{H}}, Y)$.

Theorem 6.2.16 *The* AND *of hesitant fuzzy soft UP-subalgebras of A is a hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be a hesitant fuzzy soft UP-subalgebra of A for all $i \in I$ and let $p_i \in Y_i$ for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A. Assume that $(\widetilde{\mathrm{H}}, Y)$ is the AND of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$. Then $Y = \prod_{i \in I} Y_i$ and $\widetilde{\mathrm{H}}[(p_i)_{i \in I}] = \bigcap_{i \in I} \widetilde{\mathrm{H}}_i[p_i]$. By Theorem 6.1.19, we have $\widetilde{\mathrm{H}}[(p_i)_{i \in I}] = \bigcap_{i \in I} \widetilde{\mathrm{H}}_i[p_i]$ is a hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a $(p_i)_{i \in I}$-hesitant fuzzy soft UP-subalgebra of A. Since $(p_i)_{i \in I}$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.16, we can use Theorem 6.1.20 (resp., Theorem 6.1.21, Theorem 6.1.22, Theorem 6.1.26) to prove that the AND of hesitant fuzzy soft UP-filters (resp., hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-filter (resp., a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideals) of A. By the way, we also confirm that the AND of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals) of A is not an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal) of A in general.

Definition 6.2.17 Let $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ be a family of hesitant fuzzy soft sets over a reference set X. The OR of $\{(\widetilde{\mathrm{H}}_i, Y_i) \mid i \in I\}$ is defined to be the hesitant fuzzy soft sets $(\widetilde{\mathrm{H}}, Y)$ satisfying the following properties:

(1) $Y = \prod_{i \in I} Y_i$, and

(2) $\widetilde{\mathrm{H}}[(p_i)_{i \in I}] = \bigcup_{i \in I} \widetilde{\mathrm{H}}_i[p_i]$ for all $(p_i)_{i \in I} \in Y$.

We write $\widetilde{\bigvee_{i \in I}}(\widetilde{\mathrm{H}}_i, Y_i) = (\widetilde{\mathrm{H}}, Y)$.

Theorem 6.2.18 *The* OR *of anti-hesitant fuzzy soft UP-subalgebras of A is an anti-hesitant fuzzy soft UP-subalgebra of A.*

Proof. Let $(\widetilde{\mathrm{H}}_i, Y_i)$ be an anti-hesitant fuzzy soft UP-subalgebra of A for all $i \in I$ and let $p_i \in Y_i$ for all $i \in I$. Then $\widetilde{\mathrm{H}}_i[p_i]$ is a hesitant fuzzy UP-subalgebras of A. Assume that $(\widetilde{\mathrm{H}}, Y)$ is the OR of $(\widetilde{\mathrm{H}}_i, Y_i)$ for all $i \in I$. Then $Y = \prod_{i \in I} Y_i$ and $\widetilde{\mathrm{H}}[(p_i)_{i \in I}] = \bigcup_{i \in I} \widetilde{\mathrm{H}}_i[p_i]$. By Theorem 6.1.32, we have $\widetilde{\mathrm{H}}[(p_i)_{i \in I}]$ is an anti-hesitant fuzzy UP-subalgebra of A. Therefore, $(\widetilde{\mathrm{H}}, Y)$ is a $(p_i)_{i \in I}$-anti-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

In the same way as Theorem 6.2.18, we can use Theorem 6.1.33 (resp., Theorem 6.1.34, Theorem 6.1.35 , Theorem 6.1.31) to prove that the OR of anti-hesitant fuzzy soft UP-filters (resp., anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-filter (resp., an anti-hesitant fuzzy soft UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal, a hesitant fuzzy soft strongly UP-ideal) of A. By the way, we also confirm that the OR of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals) of A is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal) of A in general.

CHAPTER VII

CONCLUSIONS

From our study, we get the main results as the following:

1. A hesitant fuzzy set H is a constant hesitant fuzzy set on A if and only if the complement of H is a constant hesitant fuzzy set on A.

2. A hesitant fuzzy set H on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A.

3. Every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal of A.

4. Every anti-hesitant fuzzy UP-ideal of A is an anti-hesitant fuzzy UP-filter of A.

5. Every anti-hesitant fuzzy UP-filter of A is an anti-hesitant fuzzy UP-subalgebra of A.

6. A hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-subalgebra of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A.

7. A hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-filter of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-filter of A.

8. A hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A.

9. Let H be a hesitant fuzzy set on A. Then the following statements are equivalent:

   (1) H is an anti-hesitant fuzzy strongly UP-ideal of A,

   (2) a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

   (3) a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$.

10. Let H be a hesitant fuzzy set on A. Then the following statements hold:

   (1) if H is an anti-hesitant fuzzy UP-subalgebra of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^{-}(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A if $L^{-}(\mathrm{H};\varepsilon)$ is nonempty, and

   (2) if Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^{-}(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A, then H is an anti-hesitant fuzzy UP-subalgebra of A.

11. Let H be a hesitant fuzzy set on A. Then the following statements hold:

   (1) if H is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^{-}(\mathrm{H};\varepsilon)$ is a UP-filter of A if $L^{-}(\mathrm{H};\varepsilon)$ is nonempty, and

   (2) if Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^{-}(\mathrm{H};\varepsilon)$ of A is a UP-filter of A, then H is an anti-hesitant fuzzy UP-filter of A.

12. Let H be a hesitant fuzzy set on A. Then the following statements hold:

   (1) if H is an anti-hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^{-}(\mathrm{H};\varepsilon)$ is a UP-ideal of A if $L^{-}(\mathrm{H};\varepsilon)$ is nonempty, and

(2) if Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A, then H is an anti-hesitant fuzzy UP-ideal of A.

13. Let H be a hesitant fuzzy set on A. Then the following statements hold:

(1) if H is an anti-hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^-(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A if $L^-(\mathrm{H};\varepsilon)$ is nonempty, and

(2) if Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A, then H is an anti-hesitant fuzzy strongly UP-ideal of A.

14. A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-subalgebra of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A.

15. A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-filter of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-filter of A.

16. A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A.

17. Let H be a hesitant fuzzy set on A. Then the following statements are equivalent:

(1) $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A,

(2) a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$, and

(3) a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A for all $\varepsilon \in \mathcal{P}([0,1])$.

18. Let H be a hesitant fuzzy set on A. Then the following statements hold:

    (1) if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^{+}(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A if $U^{+}(\mathrm{H};\varepsilon)$ is nonempty, and

    (2) if $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A.

19. Let H be a hesitant fuzzy set on A. Then the following statements hold:

    (1) if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^{+}(\mathrm{H};\varepsilon)$ is a UP-filter of A if $U^{+}(\mathrm{H};\varepsilon)$ is nonempty, and

    (2) if $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a UP-filter of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A.

20. Let H be a hesitant fuzzy set on A. Then the following statements hold:

    (1) if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^{+}(\mathrm{H};\varepsilon)$ is a UP-ideal of A if $U^{+}(\mathrm{H};\varepsilon)$ is nonempty, and

    (2) if $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A.

21. Let H be a hesitant fuzzy set on A. Then the following statements hold:

    (1) if $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $U^{+}(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A if $U^{+}(\mathrm{H};\varepsilon)$ is nonempty, and

(2) if $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^{+}(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A, then $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A.

22. If a hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-subalgebra of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A where $L^{-}(\mathrm{H};\varepsilon)$ is empty.

23. If a hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-filter of A where $L^{-}(\mathrm{H};\varepsilon)$ is empty.

24. If a hesitant fuzzy set H on A is an anti-hesitant fuzzy UP-ideal of A, then $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A where $L^{-}(\mathrm{H};\varepsilon)$ is empty.

25. A hesitant fuzzy set H on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if $E(\mathrm{H};\mathrm{h}_{\mathrm{H}}(0))$ is a strongly UP-ideal of A.

26. If a hesitant fuzzy set H on A is a hesitant fuzzy UP-subalgebra of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A where $U^{+}(\mathrm{H};\varepsilon)$ is empty.

27. If a hesitant fuzzy set H on A is a hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-filter of A where $U^{+}(\mathrm{H};\varepsilon)$ is empty.

28. If a hesitant fuzzy set H on A is a hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A where $U^{+}(\mathrm{H};\varepsilon)$ is empty.

29. A hesitant fuzzy set H on A is a hesitant fuzzy strongly UP-ideal of A if and only if $E(\mathrm{H};\mathrm{h}_{\mathrm{H}}(0))$ is a strongly UP-ideal of A.

30. If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A, then it satisfies the property:

$$(\forall p \in Y \forall x \in A)(\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.1.1}$$

31. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.1.2}$$

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of A.

32. Every p-hesitant fuzzy soft UP-filter of A is a p-hesitant fuzzy soft UP-subalgebra.

33. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft UP-filter of A if and only if it satisfies the condition (5.1.2).

34. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \tag{5.1.3}$$

Then it is a hesitant fuzzy soft UP-filter of A.

35. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \tag{5.1.4}$$

Then it is a hesitant fuzzy soft UP-filter of A.

36. Every p-hesitant fuzzy soft UP-ideal of A is a p-hesitant fuzzy soft UP-filter.

37. If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of A, then it satisfies the condition (5.1.3).

38. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft UP-ideal of A if and only if it satisfies the condition (5.1.4).

39. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.1.5)$$

Then it is a hesitant fuzzy soft UP-ideal of A.

40. Every p-hesitant fuzzy soft strongly UP-ideal of A is a p-hesitant fuzzy soft UP-ideal.

41. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft strongly UP-ideal of A if and only if it satisfies the condition (5.1.5).

42. Let $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A such that $\emptyset \neq N \subseteq Y$. Then the following statements are hold:

    (1) if $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A, then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A, and

    (2) there exists $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A.

43. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.

44. If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, then it satisfies the property:

$$(\forall p \in Y \forall x \in A)(\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.2.1}$$

45. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{5.2.2}$$

Then it is an anti-hesitant fuzzy soft UP-subalgebra of A.

46. Every p-anti-hesitant fuzzy soft UP-filter of A is a p-anti-hesitant fuzzy soft UP-subalgebra.

47. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft UP-filter of A if and only if it satisfies the condition (5.2.2).

48. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \tag{5.2.3}$$

Then it is an anti-hesitant fuzzy soft UP-filter of A.

49. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \tag{5.2.4}$$

Then it is an anti-hesitant fuzzy soft UP-filter of A.

50. Every p-anti-hesitant fuzzy soft UP-ideal of A is a p-anti-hesitant fuzzy soft UP-filter.

51. If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A, then it satisfies the condition (5.2.3).

52. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft UP-ideal of A if and only if it satisfies the condition (5.2.4).

53. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:

$$(\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)). \quad (5.2.5)$$

Then it is an anti-hesitant fuzzy soft UP-ideal of A.

54. Every p-anti-hesitant fuzzy soft strongly UP-ideal of A is a p-anti-hesitant fuzzy soft UP-ideal.

55. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if it satisfies the condition (5.2.5).

56. Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A and N be a nonempty subset of Y. Then the following statements are hold:

    (1) if $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal) of A, then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal) of A, and

    (2) there exists $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra (resp., anti-hesitant fuzzy soft

UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal) of A.

57. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.

58. The intersection of hesitant fuzzy strongly UP-ideals (resp., anti-hesitant fuzzy strongly UP-ideals) of A is a hesitant fuzzy strongly UP-ideals (resp., an anti-hesitant fuzzy strongly UP-ideal) of A.

59. The union of hesitant fuzzy strongly UP-ideals (resp., anti-hesitant fuzzy strongly UP-ideals) of A is a hesitant fuzzy strongly UP-ideals (resp., an anti-hesitant fuzzy strongly UP-ideal) of A.

60. The hesitant intersection of hesitant fuzzy UP-subalgebras (resp., hesitant fuzzy UP-filters, hesitant fuzzy UP-ideals, hesitant fuzzy strongly UP-ideals, anti-hesitant fuzzy strongly UP-ideals) of A is a hesitant fuzzy UP-subalgebra (resp., a hesitant fuzzy UP-filter, a hesitant fuzzy UP-ideal, a hesitant fuzzy strongly UP-ideal, an anti-hesitant fuzzy strongly UP-ideal) of A.

61. The hesitant union of anti-hesitant fuzzy UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals, anti-hesitant fuzzy strongly UP-ideals, hesitant fuzzy strongly UP-ideals) of A is an anti-hesitant fuzzy UP-subalgebra (resp., an anti-hesitant fuzzy UP-filters, an anti-hesitant fuzzy UP-ideals, an anti-hesitant fuzzy strongly UP-ideals, a hesitant fuzzy strongly UP-ideals) of A.

62. The restricted union of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals, anti-hesitant fuzzy strongly UP-ideals, hesitant fuzzy strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-subalgebra (resp., an anti-hesitant fuzzy UP-filters,

an anti-hesitant fuzzy UP-ideals, an anti-hesitant fuzzy strongly UP-ideals, a hesitant fuzzy strongly UP-ideals) of A.

63. The union of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-subalgebra (resp., an anti-hesitant fuzzy soft UP-filter, an anti-hesitant fuzzy soft UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal, a hesitant fuzzy soft strongly UP-ideal) of A.

64. The intersection of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-subalgebra (resp., a hesitant fuzzy soft UP-filter, a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal) of A.

65. The extended intersection of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-subalgebra (resp., a hesitant fuzzy soft UP-filter, a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal) of A.

66. The AND of hesitant fuzzy soft UP-subalgebras (resp., hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals) of A is a hesitant fuzzy soft UP-subalgebra (resp., a hesitant fuzzy soft UP-filter, a hesitant fuzzy soft UP-ideal, a hesitant fuzzy soft strongly UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal) of A.

67. The OR of anti-hesitant fuzzy soft UP-subalgebras (resp., anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft strongly UP-ideals) of A is an anti-hesitant fuzzy soft UP-subalgebra (resp., an anti-hesitant fuzzy soft UP-filter, an anti-hesitant fuzzy soft UP-ideal, an anti-hesitant fuzzy soft strongly UP-ideal, a hesitant fuzzy soft strongly UP-ideal) of A.

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Publications

Articles

1. **Mosrijai, P.** and Iampan, A. (2018). Anti-type of hesitant fuzzy sets on UP-algebras. Eur. J. Pure Appl. Math., 11(4), 976 - 1002.

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1. **Mosrijai, P.** (May 23 - 24, 2019). A new branch of bialgebraic structures: UP-bialgebras. In The 11 National Science Research Conference, Srinakharinwirot University, Bangkok, Thailand.
2. **Mosrijai, P.** (May 24 - 25, 2018). Hesitant fuzzy soft set over UP-algebras. In The 10th National Science Research Conference, Mahasarakham University, Mahasarakham, Thailand.
3. **Mosrijai, P.** (May 25 - 26, 2017). Hesitant fuzzy set on UP-algebras. In The 9th National Science Research Conference, Burapha University, Chon Buri, Thailand.