NOVEL HYBRID ITERATIVE METHODS AND ITS CONVERGENCE ANALYSIS

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บทคัดย่อ

กระบวนการทำซ้ำจุดตรึงเป็นหนึ่งในเทคนิคที่ใช้แก้สมการไม่เชิงเส้น ทฤษฎีนี้ถูกนำไปประยุกต์ใช้ในหลายสาขา เช่น คณิตศาสตร์ประยุกต์ ชีววิทยา เคมี เศรษฐศาสตร์ วิศวกรรมศาสตร์ และทฤษฎีเกม ในงานนี้เราสนใจที่จะศึกษาจุดตรึงและวิธีการประมาณค่าจุดตรึงสำหรับการส่งแบบไม่เชิงเส้น โดยลำดับการทำซ้ำถูกสร้างขึ้นตามขั้นตอนวิธีใหม่ด้วยเทคนิคเฉื่อยและวิธีการทำซ้ำผสมแบบใหม่ในปริภูมิบานาค

วัตถุประสงค์แรกของวิทยานิพนธ์นี้ คือ การพิสูจน์ว่าลำดับที่เกิดจากขั้นตอนวิธีแบบใหม่ด้วยเทคนิคเฉื่อยนั้นลู่เข้าอย่างเข้มไปยังจุดตรึงของการส่งแบบไม่ขยายในปริภูมิบานาคนูนแบบเอกรูปค่าจริง ภายใต้การ์โตนอร์มที่สามารถหาอนุพันธ์ได้แบบเอกรูป นอกจากนี้ยังได้ค่าศูนย์ของการส่งแบบแอคครีทีฟ ขั้นตอนวิธีที่นำเสนอนี้ได้ถูกนำไปทดสอบด้วยการจำลองเชิงตัวเลขใน MATLAB ผลการจำลองแสดงให้เห็นว่าอัลกอริทึมลู่เข้าภายใต้เงื่อนไขที่เหมาะสม แสดงให้เห็นถึงประสิทธิภาพของอัลกอริทึมที่ได้นำเสนอ

วัตถุประสงค์ที่สอง เป็นการนำเสนอและศึกษาวิธีการทำซ้ำแบบใหม่ที่เรียกว่า วิธีการทำซ้ำผสมแบบปิการ์ SP (เรียกย่อว่า PSPHM) วิธีการทำซ้ำใหม่นี้เป็นการผสมระหว่างวิธีการทำซ้ำของปิการ์และวิธีการทำซ้ำแบบ SP เราได้ทำการเปรียบเทียบอัตราการลู่เข้าระหว่างวิธีการทำซ้ำที่นำเสนอและวิธีการทำซ้ำอื่น ๆ ที่มีอยู่ โดยให้ตัวอย่างเชิงตัวเลข โดยเฉพาะผลลัพธ์หลักได้แสดงให้เห็นว่า PSPHM ลู่เข้าเร็วกว่า วิธีการทำซ้ำแบบนูร์ และ วิธีการทำซ้ำแบบ SP ในความหมายของเบอร์เลนเด นอกจากนี้ เรายังให้ผลลัพธ์เสถียรสำหรับวิธีการทำซ้ำที่พัฒนาใหม่นี้ ยิ่งไปกว่านั้นยังได้นำวิธีการนำเสนอไปใช้ในการสร้างภาพของโพลิโนมิโอกราฟ

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ABSTRACT

The fixed point iterative procedure is one of the techniques employed to solve nonlinear equations. This theory has found applications beyond mathematics, spanning fields such as Applied Mathematics, Biology, Chemistry, Economics, Engineering, and Game Theory. We interest in inverstigating the fixed point and approximate iterative approaches for nonlinear mappings. The sequence was created iteratively by a novel algorithm with an inertial technique, and a new hybrid iterative method is also discussed in Banach spaces.

The first purpose of this dissertation is to prove that a novel algorithm with an inertial approach, used to generate an iterative sequence, strongly converges to a fixed point of a nonexpansive mapping in a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Furthermore, zeros of accretive mappings are obtained. The proposed algorithm has been implemented and tested via numerical simulation in MATLAB. The simulation results show that the algorithm converges to the optimal configurations and shows the effectiveness of the proposed algorithm.

The second purpose is to introduce and study a new fixed point iterative method named Picard-SP hybrid iterative method (PSPHM for short). This new iterative process can be seen as a hybrid of Picard and SP iterative processes. We also compare the rate of convergence between the proposed iteration and some other iteration processes in the literature via a numerical example. Specifically, our main result shows that PSPHM converges faster than Noor and SP iterations in Berinde's sense. Moreover, we also established a stable result for our newly developed iterative process. As an application, we apply the proposed method to the visualization of polynomiographs.

The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

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CHAPTER I

INTRODUCTION

Let $\mathcal{C}$ be a nonempty closed and convex subset of a real Banach space $\mathcal{B}$ with dual $\mathcal{B}^*$. Let $J : \mathcal{B} \to 2^{\mathcal{B}^*}$ denote the normalized duality mapping given by

$$J(v) = \left\{ \varkappa \in \mathcal{B}^* : \langle v, \varkappa \rangle = \|v^2\| = \|\varkappa^2\| \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing (see for example [24]). J is single valued if $\mathcal{B}^*$ is strictly convex and in what follows we denote single valued normalized duality mapping by J. A Banach space $\mathcal{B}$ is said to be uniformly convex [5, 84] if, for any sequences $\{v_m\}$ and $\{\wp_m\}$ in $\mathcal{B}$ with $\|v_m\| = \|\wp_m\| = 1$ and $\lim_{m\to\infty} \|v_m + \wp_m\| = 2$ imply $\lim_{m\to\infty} \|v_m - \wp_m\| = 0$. The modulus of smoothness $\rho_{\mathcal{B}}(\cdot)$ of $\mathcal{B}$ is the function $\rho_{\mathcal{B}} : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_{\mathcal{B}}(\tau) = sup \left\{ \frac{1}{2} \left(\|v + \wp\| + \|v - \wp\| \right) - 1 : v, \wp \in \mathcal{B}, \|v\| = 1, \|\wp\| \le \tau \right\}.$$

It is well known that $\mathcal{B}$ is uniformly smooth if and only if $\frac{\rho_{\mathcal{B}}(\tau)}{\tau} \to 0$, as $\tau \to 0$. Let $q > 1$ be a real number. A Banach space $\mathcal{B}$ is said to be q-uniformly smooth if there exists a positive constant K_q such that $\rho_{\mathcal{B}}(\tau) \le K_q \tau^q$ for any $\tau > 0$. It is obvious that q-uniformly smooth Banach space must be uniformly smooth. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to be $L - Lipschitzian$ if there exists $L \ \geq 0$ such that

$$\|\mathcal{T}v - \mathcal{T}\wp\| \le L\|v - \wp\|, \ \forall v, \wp \in \mathcal{C}.$$

$\mathcal{T}$ is said to be a contraction if $L \in [0, 1)$ and $\mathcal{T}$ is said to be nonexpansive if

$L = 1$ (see [5, 10, 18, 24, 64, 84]).

We are interested in formulating a numerical method for solving the fixed point problem

$$\text{find } v \in \mathcal{C} \text{ such that } v = \mathcal{T}(v), \tag{1.0.1}$$

where $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a nonexpansive mapping. We consider $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and designate $\mathcal{F}(\mathcal{T})$ by the set of all fixed points of $\mathcal{T}$, which is $\mathcal{F}(\mathcal{T}) := \{v \in \mathcal{C} \mid v = \mathcal{T}(v)\}$.

The most naive approach when looking for a fixed point of a contraction mapping $\mathcal{T} : \hbar \to \hbar$ defined on a complete metric space $(\hbar, \mathcal{L})$ has a unique fixed point, where $\mathcal{L}$ is the distance that describes the mapping $\mathcal{T}$ is the following process, also called Banach-Picard iteration,

$$v_{m+1} = \mathcal{T}(v_m), \ \forall m \geq 0, \tag{1.0.2}$$

where $v_0 \in \hbar$ is a starting point.

According to the Banach-Picard fixed point theorem, if $\mathcal{T}$ is a contraction, namely, $\mathcal{T}$ is Lipschitz continuous with modulus $\delta \in [0, 1)$, then the sequence $\{v_m\}_{m \geq 0}$ generated by (1.0.2) converges strongly to the unique fixed point of $\mathcal{T}$ with linear convergence rate.

If $\mathcal{T}$ is just nonexpansive, then this statement is no longer true. To illustrate this, it is enough to choose $\mathcal{T} = -Id$, where Id denotes the identity mapping, and $v_0 \neq 0$, in which case the Banach-Picard iteration not only fails approach a fixed point of $\mathcal{T}$.

In order to overcome the restrictive contraction assumption on $\mathcal{T}$, Krasnoselskii proposed in [54] to apply the Banach-Picard iteration (1.0.2) to the operator $\frac{1}{2}Id + \frac{1}{2}\mathcal{T}$ instead of $\mathcal{T}$. The Krasnoselskii-Mann iteration is written as follows :

$$v_{m+1} = (1-\eta_m)v_m + \eta_m \mathcal{T}(v_m),\ \forall m \geq 0, \tag{1.0.3}$$

where $\{\eta_m\}$ is a sequence in $(0,1)$. This iteration is often said to be a segmenting Mann iteration (see [42, 46, 59]) or to be of Krasnoselskii-type (see e.g., [17, 31, 32, 33, 34, 35, 47]). It was found that the sequence $\{v_m\}$ created by (1.0.3) weakly converges to a fixed point of $\mathcal{T}$ under the conditions of $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and mild assumptions imposed on $\{\eta_m\}$.

It turned out that a fundamental step in proving the convergence of the iterates of (1.0.3) is to show that $v_m - \mathcal{T}(v_m) \to 0$ as $m \to +\infty$, as it was done by Browder and Petryshyn in [21] in the constant case $\eta_m \equiv \eta \in (0,1)$. The weak convergence of the iterates was then studied in various settings in [9, 16, 42, 49, 68].

It should be noted that, even in real Hilbert spaces, all previous modifications to the Krasnoselskii-Mann method for nonexpansive mappings only provide weak convergence; for further information, see [40].

Bot *et al*. [15] recently presented a new form for Manns method to address the previously mentioned issues. Let v_0 be arbitrary in a real Hilbert space $\mathcal{H}$, $\forall m \geq 0$,

$$v_{m+1} \;=\; \eta_m v_m + \zeta_m\Big(\mathcal{T}(\eta_m v_m) - \eta_m v_m\Big). \tag{1.0.4}$$

They proved that the iterative sequence $\{v_m\}$ produced by (1.0.4) is strongly convergent using appropriate $\{\eta_m\}$ and $\{\zeta_m\}$ assumptions. Sequence $\{\zeta_m\}$, also known as the Tikhonov regularization sequence, plays a significant role in acceleration (1.0.4). Dong *et al*. [30], Fan *et al*. [38], and Polyak [71] have cited several theoretical and numerical conversations to examine strong convergence utilizing the Tikhonov regularization algorithm.

Recent years have seen the development and introduction of additional algorithms, such as the inertial algorithm initially presented by Polyak [71]. He minimized a smooth convex function by use of inertial extrapolation. It is important to note that these simple adjustments improved the efficiency and efficacy of these algorithms. Researchers have been able to study several vital applications after adopting this concept. For example, see [2, 6, 30, 38, 44, 57, 60, 80, 83, 85].

An operator $\Upsilon : D(\Upsilon) \subseteq \mathcal{B} \to R(\Upsilon) \subseteq \mathcal{B}$ is called accretive (see [24]) if for all $t > 0$ and for all $v, \wp \in D(\Upsilon)$, where $D(\Upsilon)$ denotes the domain of Υ, we have

$$\|v - \wp\| \leq \|v - \wp + t(\Upsilon v - \Upsilon \wp)\|.$$

Furthermore, Υ is accretive if and only if for each $v, \wp \in D(\Upsilon)$, there exists $j(v - \wp) \in J(v - \wp)$ such that

$$\langle \Upsilon v - \Upsilon \wp, j(v - \wp) \rangle \geq 0.$$

An accretive operetor Υ is said to be m-accretive (see for example [24]) if $R(I + e\Upsilon) = \mathcal{B}$ for all $e > 0$, where $R(I+e\Upsilon)$ is the range of $(I+e\Upsilon)$. Υ is said to satisfy the range condition if $\overline{D(\Upsilon)} \subseteq R(I + e\Upsilon)$ for all $e > 0$, where $\overline{D(\Upsilon)}$ is the closure of the domain of Υ. Moreover, if Υ is accretive [27], then $J_\Upsilon : R(I + \Upsilon) \to D(\Upsilon)$, which defined by $J_\Upsilon = (I + \Upsilon)^{-1}$ is a single-valued nonexpansive and $\mathcal{F}(J_\Upsilon) = N(\Upsilon)$, where $N(\Upsilon) = \{v \in D(\Upsilon) : 0 \in \Upsilon v\}$ and $\mathcal{F}(J_\Upsilon) = \{v \in \mathcal{B} : J_\Upsilon v = v\}$.

Browder [19] and Kato [52] independently introduced the accretive operators. Due to their close relation to the existence theory for nonlinear equations of evolving in Banach spaces, the study of such mappings is very fascinating.

Under suitable Banach spaces, accretive operators play a crucial role in many physically relevant situations that may be characterized as initial boundary value problems as follows:

$$\frac{d\mu}{d\tau} + \Upsilon\mu = 0, \ \mu(0) = \mu_0. \tag{1.0.5}$$

Many embedded models of evolution equations exist, including the Schrodinger, heat, and wave equations [72]. According to Browder [19], (1.0.5) has a solution if Υ is locally Lipschitzian and accretive on $\mathcal{B}$. He also proved that Υ is m-accretive and there is a solution to the equation below

$$\Upsilon\mu = 0. \tag{1.0.6}$$

Ray [72] uses the fixed point theory of Caristi [23] to elegantly and precisely improve Browder's conclusions. Robert and Martin [77] show that the problem (1.0.5) is solved in the space $\mathcal{B}$ if Υ is continuous and accretive. Utilizing this result, Martin [61] proved that if Υ is continuous and accretive, then Υ is m-accretive.

See Browder [20] and Deimling [28] for further information on the theorems for zeros of accretive operators.

One should note that, if μ is independent of τ in (1.0.5), then $\frac{d\mu}{d\tau} = 0$. Because of this, (1.0.5) simplifies to (1.0.6), which solution illustrates the problem's stable or equilibrium state. This in turn is tremendously fascinating in a variety of beautiful applications, including, but not limited to, economics, physics, and ecology. Significant efforts have been undertaken to solve (1.0.6) when Υ is accretive. Researchers were interested in investigating the fixed point and approximate iterative approaches for zeros of m-accretive mappings since Υ, in general, is nonlinear and there is no known process to discover a close solution to this equation. As a result, research in the field has flourished up to the present.

Some of the related work can be found in [86, 88] and the references therein.

Based on the previous research, in the first part of this dissertation, the sequence was created iteratively by a novel algorithm with an inertial technique, and a strong convergence using the proposed algorithm is also discussed in a real uniformly convex Banach space with Gâteaux differentiable norm. In addition, we find zeros of accretive mappings. Moreover, a numerical example is presented to illustrate the behavior of our algorithm.

Banach [7] outlined a very basic idea of contraction mapping and proved the well known Banach contraction principle. This result is the basis of fixed point theory, which guarantees not only the fixed point of contraction mapping but also the uniqueness of the fixed point. Browder [18], Gohde [41], and Kirk [53] extended the idea of Banach and introduced new research dimensions in the field of fixed point theory.

In the second part of this dissertation, we will denote C to be a nonempty closed convex subset of a real Banach space X.

Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$ the asymptotic radius of $\{x_n\}$ at x is the number $r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|$. The real number

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

is called the asymptotic radius of $\{x_n\}$ relative to C and finally the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\},$$

is called the asymptotic center of $\{x_n\}$ relative to C. It has been proven by Edelstein [35] that, for a nonempty, closed, and convex subset of a uniformly convex Banach space and for each bounded sequence $\{x_n\}$, the set $A(C, \{x_n\})$ is a singleton.

Following Banach's work in 1922, various schemes for approximating fixed points of contractive maps emerged. We will mention a few works directly related to the proposed scheme.

In 1953, Mann [59] defined the following one step iteration process for sequence $\{x_n\}$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \geq 1, \tag{1.0.7}$$

where the sequence $\{\alpha_n\}$ belongs to $(0,1)$. Mann showed that after taking $\alpha_n = 1$ in (1.0.7), it converts into Picard's iterative process. Thus they claim that Mann iteration is generalization of Picard iteration process.

Ishikawa [48] extended Mann's result by introducing the two-step iterative scheme: $x_1 \in C$,

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \ n \geq 1, \end{aligned} \tag{1.0.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in [0,1)$. The goal was to achieve convergence for Lipschitzian pseudo-contractive maps where the Mann iterative algorithm fails to converge. If $\beta_n = 0$ for all $n \geq 1$ in equation (1.0.8), the Ishikawa iterative scheme reduces to the Mann iterative scheme (1.0.7).

In 2000, Noor [63] introduced a three-step iterative scheme (also known as the Noor Iteration). This scheme extends the results of Banach [7], Mann [59], and Ishikawa [48]. The scheme is defined as follows: $t_1 \in C$,

$$v_n = (1-\gamma_n)t_n + \gamma_n T t_n,$$

$$u_n = (1 - \beta_n)t_n + \beta_n T v_n,$$
$$t_{n+1} = (1 - \alpha_n)t_n + \alpha_n T u_n, \ n \geq 1, \tag{1.0.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \in [0, 1)$. Similarly, if $\gamma_n = 0$ for all $n \geq 1$, then (1.0.9) reduces to (1.0.8).

In 2011, Phuengrattana and Suantai [70] introduced the following new three-step iteration process known as the SP-iteration: $u_1 \in C$,

$$\begin{aligned} w_n &= (1 - \gamma_n)u_n + \gamma_n T u_n, \\ v_n &= (1 - \beta_n)w_n + \beta_n T w_n, \\ u_{n+1} &= (1 - \alpha_n)v_n + \alpha_n T v_n, \ n \geq 1, \end{aligned} \tag{1.0.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in (0,1).

In 2019, Kanayo Stella and Husdson (see [36]) gave the idea of a hybrid iteration process called Picard−Noor iteration process, which generates the sequence $\{x_n\}$ given as: $x_1 \in C$,

$$\begin{aligned} w_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \\ z_n &= (1 - \beta_n)x_n + \beta_n T w_n, \\ y_n &= (1 - \alpha_n)x_n + \alpha_n T z_n, \\ x_{n+1} &= T y_n, \ n \geq 1, \end{aligned} \tag{1.0.11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in (0,1).

The hybrid iterative scheme (1.0.11) motivate us to introduce a new fixed point iterative method, named Picard-SP hybrid iterative method (PSPHM for short). This new iterative process can be seen as a hybrid of Picard and SP iterative processes (1.0.10), respectively. The scheme is defined as follow: $x_1 \in C$,

$$\begin{aligned} w_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ z_n &= (1-\beta_n)w_n + \beta_n T w_n, \\ y_n &= (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} &= T y_n, \ n \geq 1, \end{aligned} \tag{1.0.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in (0,1).

We show that our Picard-SP hybrid method (1.0.12) gives faster rate of convergence than existing iterative processes (1.0.9) and (1.0.10). We show this by comparison tables using the MATLAB programming. Moreover, we also show that our iteration (1.0.12) is T-Stable. Furthermore, we show the use of the proposed method to generate polynomiographs.

As reviewed, it is therefore the main objectives in this dissertation to introduce and study two type of iterative procedures for given mappings in Banach spaces. Furthermore, we then establish strong convergence theorems under some mild conditions in Banach spaces. And finally, we also present some examples, using MATHLAB programing. We have also given a graphical representation for this.

The results presented here extend and improve some related results in the literature.

CHAPTER II

PRELIMINARIES

2.1 Metric spaces and Banach spaces

Now, we recall some well known concepts and results.

Definition 2.1.1. [55] A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Definition 2.1.2. [55] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n \to \infty} x_n = x \quad \text{or, simple} \quad x_n \to x$$

we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 2.1.3. [55] A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if for every $\epsilon > 0$ there is an $N(\epsilon) \in N$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$.

Definition 2.1.4. [55] A metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.5. [55] Every convergent sequence in a matric space is a Cauchy sequence.

Theorem 2.1.6 *[62] Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.*

Definition 2.1.7. [62] Let X be a matric space and A be any nonempty subset of X. For each x in X, the distance $d(x, A)$ from x to A is $\inf\{d(x, y)|y \in A\}$.

Definition 2.1.8. [62] Let X be a linear space (or vector space). A norm on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:
(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\alpha x\| = |\alpha|\|x\|$,
(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).
The ordered pair $(X, \|\cdot\|)$ is called a normed space or normed vector space or normed linear space.

Definition 2.1.9. [62] Let X be normed space. The metric induced by the norm of X is the metric d on X defined by the formula $d(x, y) = \|x-y\|$ for all $x, y \in X$. The norm topology of X is the topology obtained from this metric.

Definition 2.1.10. [62] A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a **Banach space** or B-space or complete normed space if its norm is a Banach norm.

A real normed linear space $\mathcal{B}$ is said to have a Gâteaux differentiable norm if the limit

$$\lim_{\tau \to \infty} \frac{\|v + \tau\wp\| - \|v\|}{\tau},$$

exists for all $v, \wp \in \aleph$, where $\aleph$ denotes the unit sphere of $\mathcal{B}$ (i.e., $\aleph = \{v \in \mathcal{B} : \|v\| = 1\}$), in this case $\mathcal{B}$ is called smooth. It is also said to be uniformly smooth if the limit is attained uniformly for $v, \wp \in \aleph$, and $\mathcal{B}$ is said to have a uniformly Gâteaux differentiable norm.

If $\mathcal{B}$ is smooth, it is clear that every duality mapping on $\mathcal{B}$ is a single-valued mapping. If $\mathcal{B}$ has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $\mathcal{B}$.

Let Δ be a nonempty, closed, convex and bounded subset of a real Banach space $\mathcal{B}$ and the diameter of Δ defined by $d(\Delta) = \sup\{\|v - \wp\|, v, \wp \in \Delta\}$. The Chebyshev radius of Δ given by $w(\Delta) = \inf\{w(v, \wp), v \in \Delta\}$, where $v \in \Delta$,

$$w(v, \Delta) = \sup\{\|v - \wp\|, \wp \in \Delta\}.$$

Bynum [22] proposed the normal structural coefficient $N(\mathcal{B})$ of $\mathcal{B}$ as follows:

$$N(\mathcal{B}) = \inf\left\{\frac{d(\Delta)}{w(\Delta)} : d(\Delta) > 0\right\}.$$

If $N(\mathcal{B}) > 1$, then $\mathcal{B}$ has a uniform normal structure.

Every space with a uniform normal structure is reflexive, which means that all uniformly convex and uniformly smooth Banach spaces have a uniform normal structure. See [24, 56] for more details.

Let now state some definitions and lemmas that will be useful in the coming theories.

Lemma 2.1.11 ([89]) *Suppose that $\mathcal{B}$ is a real uniformly convex Banach space. For arbitrary $u > 0$, $\aleph_u(0) = \{v \in \mathcal{B} : \|v\| \leq u\}$ and $\alpha \in [0, 1]$. Then there is a continuous strictly increasing convex function $r : [0, 2u] \to \mathbb{R}$, $r(0) = 0$ such that*

$$||\alpha v + (1-\alpha)\wp||^2 \leq \alpha||v||^2 + (1-\alpha)||\wp||^2 - \alpha(1-\alpha)r(||v - \wp||).$$

Lemma 2.1.12 ([28]) *Suppose that $\mathcal{B}$ is a real normed linear space. Then for any $v, \wp \in \mathcal{B}$, $j(v+\wp) \in J(v+\wp)$, we have the following inequality holds*

$$||v+\wp||^2 \leq ||v||^2 + 2\langle \wp, j(v+\wp)\rangle.$$

Lemma 2.1.13 ([18]) *Let $\mathcal{B}$ be a uniformly convex Banach space and Δ a nonempty, closed and convex subset of $\mathcal{B}$. Suppose that $\mathcal{T} : \Delta \to \Delta$ is a nonexpansive mapping with a fixed points. Let $\{v_m\}$ be a sequence in Δ such that $v_m \rightharpoonup v$ and $v_m - \mathcal{T}v_m \longrightarrow \wp$. Then $v - \mathcal{T}v = \wp$.*

Lemma 2.1.14 ([56]) *Let $\mathcal{B}$ be a Banach space with uniform normal structure and Δ a nonempty bounded subset of $\mathcal{B}$. Suppose that $\mathcal{T} : \Delta \longrightarrow \Delta$ is a uniformly L-Lipschitzian mapping with $L < N(\mathcal{B})^{\frac{1}{2}}$. If there is a nonempty bounded closed convex subset $\mathfrak{R}$ of Δ with the property (D), that is,*

$$v \in \mathfrak{R} \Rightarrow \varpi_w(v) \in \mathfrak{R},$$

then $\mathcal{T}$ has a fixed point in Δ.

Note that $\varpi_w(v) = \{\wp \in \mathcal{B} : y = weak\ \varpi - lim\mathcal{T}^{n_j}v, \exists\ n_j \to \infty\}$ here is the ϖ-limit set of $\mathcal{T}$ at v.

Lemma 2.1.15 ([81]) *Suppose that $(v_0,\ v_1, v_2,\ ...) \in l_\infty$, is so that $\delta_m v_m \leq 0$ for all Banach limits δ. If $\limsup_{m\to\infty}(v_{m+1} - v_m) \leq 0$, then $\limsup_{m\to\infty} v_m \leq 0$.*

Lemma 2.1.16 ([89]) *Let $\{e_m\}$ be a sequence of non-negative real numbers such that*

$$e_{m+1} \leq (1 - c_m)\,e_m + c_m\sigma_m + \pi_m,\ \ m \geq 1.$$

If

(i) $\{c_m\} \subset [0,1], \sum c_m = \infty, \limsup_{m\to\infty} \sigma_m \leq\ 0,$

(ii) for each $m \geq 0,\ \pi_m \geq 0, \sum \pi_m < \infty,$

then $\lim_{m\to\infty} e_m = 0$.

Lemma 2.1.17 [1] *Let* $\{x_n\}$ *be a sequence of positive real numbers which satisfies:*

$$x_{n+1} \leq (1-\mu_n)x_n, \;\; n \geq 1.$$

If $\{\mu_n\} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \mu_n = \infty$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 2.1.18 [87] *Let* $\{a_n\}$ *and* $\{b_n\}$ *be non-negative real sequences satisfying the following inequality.*

$$a_{n+1} \leq (1-c_n)a_n + b_n,$$

where $c_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} c_n = \infty$ *and* $\frac{b_n}{c_n} \to 0$ *as* $n \to \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

2.2 Rate of convergance and T-stable

Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a fixed point p of a given operator T. The sequence $\{x_n\}$ is better than $\{y_n\}$ in the sense of Rhoades [73] if

$$\|x_n - p\| \leq \|y_n - p\|$$

for all $n \in \mathbb{N}$. The definitions presented by Berinde [12] are as follows:

Definition 2.2.1. [12] Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers converging to u and v, respectively. The sequence $\{u_n\}$ is said to converge faster than $\{v_n\}$ if

$$\lim_{n\to\infty} \left| \frac{u_n - u}{v_n - v} \right| = 0.$$

Definition 2.2.2. [12] Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. Suppose that the error estimates

$$\|x_n - p\| \leq u_n \text{ for all } n \in \mathbb{N},$$

$$\|y_n - p\| \leq v_n \text{ for all } n \in \mathbb{N},$$

are available, where $\{u_n\}$ and $\{v_n\}$ are two sequences of positive numbers converging to zero. If $\{u_n\}$ converges faster than $\{v_n\}$, then $\{x_n\}$ converges faster than $\{y_n\}$ to p.

Several authors have presented the comparison of rate of convergence of various iterative processes (one can see [8, 11, 13, 14, 25, 26, 29, 43, 73, 76, 90]).

Definition 2.2.3. [12] Let $T, \widetilde{T} : C \to C$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for a fixed $\epsilon > 0$ we have

$$\left\| Tx - \widetilde{T}\widetilde{x} \right\| \leq \epsilon.$$

After the advent of computational mathematics, the iterative aspects of fixed point theory gained unprecedented attention. Following the discussion above, mathematicians recognized the importance of assessing the stability of methods used to approximate fixed points of operators before applying them. In 1967, Ostrowski [67] introduced the pioneering result on T-stability. Following Ostrowskis pioneering result, subsequent researchers made significant contributions to the study of stability. Notably, Harder and Hicks [45] in 1988 and Rhoades [74, 75]. In addition to Ostrowskis work, other notable contributions on stability came from Osilike [65] in 1995, Osilike and Udemene [66]in 1999, and Berinde [10] in 2002. These researchers provided clear explanations of stability concepts and introduced simpler approaches compared to Harder and Hicks [45]. The following definition is credited to Harder and Hicks [45].

Definition 2.2.4. [10] Let X be a Banach space and, $T : X \to X$ a self map, $x_0 \in X$ and the iteration procedure defined by

$$x_{n+1} = f(T, x_n), \ n = 0, 1, 2, \ldots \tag{2.2.1}$$

such that the generated sequence $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and the set

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|$$

for $n = 0, 1, 2, \ldots$, then the iteration process (2.2.1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

CHAPTER III

MAIN RESULTS

3.1 A novel algorithm with an inertial technique for fixed points of nonexpansive mappings in Banach spaces

In this section, we summarize notations and lemmas which play significant role in convergence analysis of our algorithm.

Theorem 3.1.1 *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{B}$ which has uniformly Gâteaux differentiable norm and $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ a nonexpansive mapping such that $\mathcal{F}(\mathcal{T})\neq\emptyset$. Consider that the following assumptions hold:*

(i) $\lim_{m\to\infty}\xi_m = 0$, $\lim_{m\to\infty}\sigma_m = 0$, $\sum_{m=1}^{\infty}\sigma_m = \infty$, $\xi_m, \sigma_m \in (0,1), \rho_m \in [l_1, l_2] \subset (0,1)$,

(ii) $\pi_m \geq 0$, $\forall m \in \mathbb{N}$ *and* $\sum_{m=1}^{\infty}\pi_m < \infty$.

For arbitrary $\nu_0, \nu_1 \in \mathcal{C}$. Let $\{\upsilon_m\}$ be the sequence generated by

$$\begin{cases} \hbar_m = \upsilon_m + \pi_m\,(\upsilon_m - \upsilon_{m-1}), \\ \psi_m = (1-\xi_m)\,(1-\sigma_m)\,\hbar_m, \\ \upsilon_{m+1} = (1-\rho_m\ )\psi_m + \rho_m\mathcal{T}\psi_m,\ \ m \geq 1. \end{cases} \tag{3.1.1}$$

Then $\{\upsilon_m\}$ converges strongly to a point in $\mathcal{F}(\mathcal{T})$.

Proof. Let $d \in \mathcal{F}(\mathcal{T})$. Setting $\wp_m = (1-\sigma_m)\,\hbar_m$. Using (3.1.1), we have

$$\begin{aligned} \|\upsilon_{m+1} - d\| &= \|(1-\rho_m)(\psi_m - d) + \rho_m(\mathcal{T}\psi_m - d)\| \\ &\leq (1-\rho_m\ )\|\psi_m - d\| + \rho_m\|\mathcal{T}\psi_m - d\| \end{aligned}$$

$$
\begin{aligned}
&= (1-\rho_m)\|\psi_m - d\| + \rho_m\|\mathcal{T}\psi_m - d\| \\
&\leq (1-\rho_m)\|\psi_m - d\| + \rho_m\|\psi_m - d\| \\
&= \|\psi_m - d\| \\
&= \|(1-\xi_m)\wp_m - d\| \\
&= \|(1-\xi_m)(\wp_m - d) - \xi_m d\| \\
&\leq (1-\xi_m)\|\wp_m - d\| + \xi_m\|d\| \\
&= (1-\xi_m)\|(1-\sigma_m)\hbar_m - d\| + \xi_m\|d\| \\
&\leq (1-\xi_m)\big((1-\sigma_m)\|\hbar_m - d\| + \sigma_m\|d\|\big) + \xi_m\|d\| \\
&= (1-\xi_m)(1-\sigma_m)\|\hbar_m - d\| + (1-\xi_m)\sigma_m\|d\| + \xi_m\|d\| \\
&\leq (1-\xi_m)(1-\sigma_m)\|\hbar_m - d\| + (1-\xi_m)\|d\| + \xi_m\|d\| \\
&\leq (1-\sigma_m)\|\hbar_m - d\| + \|d\| \\
&\leq (1-\sigma_m)\|(\upsilon_m - d) + \pi_m(\upsilon_m - \upsilon_{m-1})\| + \|d\| \\
&\leq (1-\sigma_m)\|\upsilon_m - d\| + (1-\sigma_m)\pi_m\|\upsilon_m - \upsilon_{m-1}\| + \|d\| \\
&\leq max\{\|\upsilon_m - d\|, \|\upsilon_m - \upsilon_{m-1}\|, \|d\|\}.
\end{aligned}
$$

By mathematical induction, one can obtain

$$\|\upsilon_m - d\| \leq max\{\|\upsilon_1 - d\|, \|\upsilon_1 - \upsilon_0\|, \|d\|\}.$$

This shows that $\{\upsilon_m\}$ is bounded, so $\{\hbar_m\}$, $\{\wp_m\}$ and $\{\psi_m\}$ are also bounded. By condition (ii). This implies $\sum_{m=1}^{\infty}\pi_m\|\nu_m - \nu_{m-1}\| < \infty$. Using Lemma 2.1.11, Lemma 2.1.12 and (3.1.1), we have

$$
\begin{aligned}
\|\upsilon_{m+1} - d\|^2 &= \|(1-\rho_m)(\psi_m - d) + \rho_m(\mathcal{T}\psi_m - d)\|^2 \\
&\leq (1-\rho_m)\|\psi_m - d\|^2 + \rho_m\|\mathcal{T}\psi_m - d\|^2 - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \\
&\leq (1-\rho_m)\|\psi_m - d\|^2 + \rho_m\|\psi_m - d\|^2 - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|)
\end{aligned}
$$

$$
\begin{aligned}
&= \|\psi_m - d\|^2 - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \\
&= \|\wp_m - d\|^2 + 2\xi_m\langle \wp_m - d, j(\psi_m - d)\rangle - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \\
&\leq \|\hbar_m - d\|^2 + 2\sigma_m\langle \hbar_m - d, j(\wp_m - d)\rangle + 2\xi_m\langle \wp_m - d, j(\psi_m - d)\rangle \\
&\quad - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \\
&\leq \|v_m - d\|^2 + 2\pi_m\langle v_m - d, j(\hbar_m - d)\rangle + 2\sigma_m\langle \hbar_m - d, j(\wp_m - d)\rangle \\
&\quad + 2\xi_m\langle \wp_m - d, j(\psi_m - d)\rangle - \rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|).
\end{aligned}
$$

On the other hand, one can write

$$
\begin{aligned}
\rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \leq \|v_m - d\|^2 - \|v_{m+1} - d\|^2 + 2\pi_m\langle v_m - d, j(\hbar_m - d)\rangle \\
+ 2\sigma_m\langle \hbar_m - d, j(\wp_m - d)\rangle + 2\xi_m\langle \wp_m - d, j(\psi_m - d)\rangle.
\end{aligned} \tag{3.1.2}
$$

The boundedness of $\{v_m\}$, $\{\hbar_m\}$, $\{\wp_m\}$ and $\{\psi_m\}$ leads to there are constants $\Lambda_1, \Lambda_2, \Lambda_3 > 0$ so that for all $m \geq 1$,

$$
\langle v_m - d, j(\hbar_m - d)\rangle \leq \Lambda_1, \langle \hbar_m - d, j(\wp_m - d)\rangle \leq \Lambda_2, \langle \wp_m - d, j(\psi_m - d)\rangle \leq \Lambda_3. \tag{3.1.3}
$$

Applying (3.1.3) in (3.1.2), we have

$$
\rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \leq \|v_m - d\|^2 - \|v_{m+1} - d\|^2 + 2\pi_m\Lambda_1 + 2\sigma_m\Lambda_2 + 2\xi_m\Lambda_3. \tag{3.1.4}
$$

This implies that $\{v_m\}$ converges to d. We consider the following cases in order to achieve strong convergence:

Case (a). If the sequence $\{\|\upsilon_m - d\|\}$ is monotonically decreasing, then $\{\|\upsilon_m - d\|\}$ is convergent. We see that

$$\|\upsilon_{m+1} - d\|^2 - \|\upsilon_m - d\|^2 \to 0$$

as $m \to \infty$. By (3.1.4), we have

$$\rho_m(1-\rho_m)r(\|\mathcal{T}\psi_m - \psi_m\|) \to 0.$$

Using the property of r and $\rho_m \in [l_1, l_2] \subset (0,1)$, we have

$$\|\mathcal{T}\psi_m - \psi_m\| \to 0. \tag{3.1.5}$$

Combining (3.1.1) and (3.1.5), we find that

$$\|\upsilon_{m+1} - \psi_m\| = \rho_m(\mathcal{T}\psi_m - \psi_m) \to 0. \tag{3.1.6}$$

Using (3.1.1) and condition (i), we have

$$\|\psi_m - \wp_m\| = \xi_m\|\wp_m\| \to 0. \tag{3.1.7}$$

From (3.1.1) and condition (i) we get

$$\|\wp_m - \hbar_m\| = \sigma_m\|\hbar_m\| \to 0. \tag{3.1.8}$$

It follows from (3.1.7) and (3.1.8) that

$$\|\psi_m - \hbar_m\| \le \|\psi_m - \wp_m\| + \|\wp_m - \hbar_m\| \to 0. \tag{3.1.9}$$

From $\sum_{m=1}^{\infty} \pi_m \|\nu_m - \nu_{m-1}\| < \infty$, we get

$$\|\hbar_m - \upsilon_m\| = \pi_m \|\upsilon_m - \upsilon_{m-1}\| \to 0. \tag{3.1.10}$$

Based on (3.1.9) and (3.1.10), we can write

$$\|\psi_m - \upsilon_m\| \leq \|\psi_m - \hbar_m\| + \|\hbar_m - \upsilon_m\| \to 0. \tag{3.1.11}$$

Using (3.1.6) and (3.1.11), we have

$$\|\upsilon_{m+1} - \upsilon_m\| \leq \|\upsilon_{m+1} - \psi_m\| + \|\psi_m - \upsilon_m\| \to 0 \quad as \quad m \to \infty.$$

Using (3.1.5), (3.1.9) and (3.1.10), we have

$$\begin{aligned}
\|\mathcal{T}\upsilon_m - \upsilon_m\| &\leq \|\mathcal{T}\upsilon_m - \mathcal{T}\psi_m\| + \|\mathcal{T}\psi_m - \psi_m\| + \|\upsilon_m - \psi_m\| \\
&\leq 2\|\upsilon_m - \psi_m\| + \|\mathcal{T}\psi_m - \psi_m\| \\
&\leq 2(\|\psi_m - \hbar_m\| + \|\hbar_m - \upsilon_m\|) + \|\mathcal{T}\psi_m - \psi_m\| \to 0.
\end{aligned}$$

Since $\{\upsilon_m\}$ is bounded, there exists a subsequence $\{\upsilon_{m_b}\} \subset \{\upsilon_m\}$ such that it converges weakly to $d \in \mathcal{B}$. In addition, using Lemma 2.1.13, we have $d \in \mathcal{F}(\mathcal{T})$.

Now, we prove that

$$\limsup_{m\to\infty} \langle -d, j(\wp_m - d)\rangle \leq 0.$$

Suppose that $\chi : \mathcal{B} \to \mathbb{R}$ is given by

$$\chi(\upsilon) = \delta_m \|\wp_m - \upsilon\|^2, \quad \forall \upsilon \in \mathcal{B}.$$

Then, $\chi(\upsilon) \to \infty$ as $\|\upsilon\| \to \infty$, χ is convex and continuous. Since $\mathcal{B}$ is reflexive, then there exists $\wp^* \in \mathcal{B}$ such that $\chi(\wp^*) = \min_{a\in\mathcal{B}} \chi(a)$. Hence, the set $\hat{\mathfrak{R}} \neq \emptyset$, where

$$\hat{\mathfrak{R}} = \left\{ v \in \mathcal{B} : \chi(v) = \min_{a\in\mathcal{B}} \chi(a) \right\}.$$

It following from $\lim_{m\to\infty} \|\mathcal{T}\psi_m - \psi_m\| = 0$ *and* $\lim_{m\to\infty} \|\psi_m - \wp_m\| = 0$ that

$$\begin{aligned}
\|\mathcal{T}\wp_m - \wp_m\| &\leq \|\mathcal{T}\wp_m - \mathcal{T}\psi_m\| + \|\mathcal{T}\psi_m - \psi_m\| + \|\psi_m - \wp_m\| \\
&\leq \|\wp_m - \psi_m\| + \|\mathcal{T}\psi_m - \psi_m\| + \|\psi_m - \wp_m\| \\
&\to 0 \quad (as\ m \to \infty).
\end{aligned}$$

Since $\lim_{m\to\infty} \|\mathcal{T}\wp_m - \wp_m\| = 0$. It follows from induction that $\lim_{m\to\infty} \|\mathcal{T}^n\wp_m - \wp_m\| = 0$ for all $n \geq 1$. Thus, using Lemma 2.1.14, if $v \in \mathfrak{R}$ and $\wp = \varpi - \lim_{j\to\infty} \mathcal{T}^{n_j} v$, then from weak lower semicontinuity of χ and $\lim_{m\to\infty} \|\mathcal{T}\wp_m - \wp_m\| = 0$. Then we get

$$\begin{aligned}
\chi(\wp) &\leq \liminf_{j\to\infty} \chi(\mathcal{T}^{n_j} v) \leq \limsup_{n\to\infty} \chi(\mathcal{T}^n v) \\
&= \limsup_{n\to\infty} (\delta_m \|\wp_m - \mathcal{T}^n v\|^2) \\
&= \limsup_{n\to\infty} (\delta_m \|\wp_m - \mathcal{T}\wp_m + \mathcal{T}\wp_m - \mathcal{T}^n v\|^2) \\
&\leq \limsup_{n\to\infty} (\delta_m \|\mathcal{T}\wp_m - \mathcal{T}^n v\|^2) \\
&\leq \limsup_{n\to\infty} (\delta_m \|\wp_m - v\|^2) = \chi(v) = \inf_{a\in\mathcal{B}} \chi(a).
\end{aligned}$$

Hence, $\wp^* \in \hat{\mathfrak{R}}$. It follows from Lemma 2.1.14 that $\mathcal{T}$ has a fixed point in $\hat{\mathfrak{R}}$, and so $\hat{\mathfrak{R}} \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$. Without losing the general case, as a particular instance, suppose that $\wp^* = d \in \hat{\mathfrak{R}} \cap \mathcal{F}(\mathcal{T})$. Consider $\tau \in (0, 1)$. Then it is easy to see that $\chi(d) \leq \chi(d - \tau d)$ with the helping of Lemma 2.1.12, we have

$$\|\wp_m - d + \tau d\|^2 \leq \|\wp_m - d\|^2 + 2\tau \langle d, j(\wp_m - d + \tau d)\rangle.$$

By the properties of χ, we can write

$$\frac{1}{\delta_m}\chi(d-\tau d) \leq \frac{1}{\delta_m}\chi(d) + 2\tau\langle d, j(\wp_m - d + \tau d)\rangle.$$

By arranging the above inequality, we have

$$2\tau\delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \leq \chi(d) - \chi(d-\tau d) \leq 0.$$

This leads to

$$\delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \leq 0.$$

In addition,

$$\begin{aligned}\delta_m\langle -d, j(\wp_m - d)\rangle &\leq \delta_m\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle + \delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \\ &\leq \delta_m\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle. \qquad (3.1.12)\end{aligned}$$

Since the normalized duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $\mathcal{B}$, we have, as $\tau \to 0$ and for fixed n,

$$\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle$$

$$\leq \langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_m - d + \tau d)\rangle \to 0.$$

Thus, for each $\epsilon > 0$, there is $\varsigma_\epsilon > 0$ such that for all $\tau \in (0, \varsigma_\epsilon)$,

$$\langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_m - d + \tau d)\rangle < \epsilon.$$

Thus,

$$\delta_m\langle -d, j(\wp_m - d)\rangle - \delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \leq \epsilon.$$

Since ϵ is an arbitrary. Using (3.1.8), we obtain

$$\delta_m\langle -d, j(\wp_m - d)\rangle \leq 0.$$

By triangle inequality, we have

$$\|\wp_{m+1} - \wp_m\| \leq \|\wp_{m+1} - \hbar_{m+1}\| + \|\hbar_{m+1} - \upsilon_{m+1}\| + \|\upsilon_{m+1} - \psi_m\| + \|\psi_m - \wp_m\|.$$

Using (3.1.4), (3.1.5), (3.1.6) and (3.1.8), we have

$$\lim_{m\to\infty} \|\wp_{m+1} - \wp_m\| = 0.$$

Again, since the normalized duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $\mathcal{B}$, we have

$$\lim_{m\to\infty} (\langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_{m+1} - d)\rangle) = 0.$$

Using Lemma 2.1.15, we have

$$\limsup_{m\to\infty}\langle -d, j(\wp_m - d)\rangle \leq 0.$$

From (3.1.1) we obtain

$$\begin{aligned}\psi_m &= (1-\xi_m)\wp_m \\ &= (1-\xi_m)(1-\sigma_m)\hbar_m \\ &\leq (1-\sigma_m)\hbar_m.\end{aligned}$$

Thus

$$\|\psi_m - d\|^2 \leq \|(1-\sigma_m)\hbar_m - d\|^2$$
$$\leq \|(1-\sigma_m)(\hbar_m - d) - \sigma_m d\|^2. \tag{3.1.13}$$

Since

$$\|\wp_m - d\|^2 = \|(1-\sigma_m)(\hbar_m - d) - \sigma_m d\|^2,$$

using (3.1.1), (3.1.13), Lemma 2.1.12 and $\sum_{m=1}^{\infty} \pi_m \|\nu_m - \nu_{m-1}\| < \infty$, we have

$$\begin{aligned}
\|\upsilon_{m+1} - d\|^2 &= \|(1-\rho_m)(\psi_m - d) + \rho_m(\mathcal{T}\psi_m - d)\|^2 \\
&\leq (1-\rho_m)\|\psi_m - d\|^2 + \rho_m\|\mathcal{T}\psi_m - d\|^2 \\
&\leq \|\psi_m - d\|^2 \\
&\leq \|(1-\sigma_m)(\hbar_m - d) - \sigma_m d\|^2 \\
&= (1-\sigma_m)\|\hbar_m - d\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&\leq (1-\sigma_m)\|(\upsilon_m - d) + \pi_m(\upsilon_m - \upsilon_{m-1})\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&\leq (1-\sigma_m)\|\upsilon_m - d\|^2 + 2\pi_m\langle \upsilon_m - \upsilon_{m-1}, j(\hbar_m - d)\rangle + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&= (1-\sigma_m)\|\upsilon_m - d\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle. \qquad (3.1.14)
\end{aligned}$$

Applying Lemma 2.1.16, we conclude that $\{\upsilon_m\}$ converges strongly to d.

Case (b) Suppose the sequence $\{\|\upsilon_m - d\|\}$ is not monotonically decreasing. Let $\Xi_m = \|\upsilon_m - d\|^2$. Suppose that $\Pi : \mathbb{N} \to \mathbb{N}$ is defined by

$$\Pi(m) = max\{\hbar \in \mathbb{N} : \hbar \leq m,\ \Xi_\hbar \leq \Xi_{\hbar+1}\}.$$

Obviously, Π is a non-decreasing sequence so that $\lim_{m\to\infty} \Pi(m) = \infty$ and $\Xi_{\Pi(m)} \leq$

$\Xi_{\Pi(m)+1}$ for $m \geq m_0$ (for some m_0 large enough). Using (3.1.4), we have

$$\begin{aligned}\rho_{\Pi_{(m)}}\Big(1-\rho_{\Pi_{(m)}}\Big)r\Big(\|\mathcal{T}\psi_{\Pi_{(m)}}-\psi_{\Pi_{(m)}}\|\Big) \leq\ & \|\upsilon_{\Pi(m)}-d\|^2-\|\upsilon_{\Pi(m+1)}-d\|^2+2\pi_{\Pi(m)}\Lambda_1\\ &+2\sigma_{\Pi(m)}\Lambda_2+2\xi_{\Pi_{(m)}}\Lambda_3\\ =\ & \Xi_{\Pi(m)}-\Xi_{\Pi(m)+1}+2\pi_{\Pi(m)}\Lambda_1+2\sigma_{\Pi(m)}\Lambda_2\\ &+2\xi_{\Pi_{(m)}}\Lambda_3\\ \leq\ & 2\pi_{\Pi(m)}\Lambda_1+2\sigma_{\Pi(m)}\Lambda_2+2\xi_{\Pi_{(m)}}\Lambda_3\\ &\to 0\ as\ m\to\infty.\end{aligned}$$

In addition, we get

$$\|\mathcal{T}\psi_{\Pi(m)}-\psi_{\Pi(m)}\|\to 0\ as\ m\to\infty.$$

Using the same circumstances as in Case (a), we can show that $\upsilon_{\Pi(m)} \rightharpoonup d$ as $\Pi(m)\to\infty$ and $\limsup_{\Pi(m)\to\infty}\langle -d, j(\wp_{\Pi(m)}-d)\rangle \leq 0$. For all $m \geq m_0$, we obtain by (3.1.14) that

$$0\leq \Big\|\upsilon_{\Pi(m)+1}-d\Big\|^2-\Big\|\upsilon_{\Pi(m)}-d\Big\|^2\leq \sigma_{\Pi(m)}\Big[2\langle -d, j(\wp_{\Pi(m)}-d)\rangle-\Big\|\upsilon_{\Pi(m)}-d\Big\|^2\Big].$$

This implies that

$$\Big\|\upsilon_{\Pi(m)}-d\Big\|^2\leq 2\langle -d, j(\wp_{\Pi(m)}-d)\rangle.$$

Since $\limsup_{\Pi(m)\to\infty}\langle -d, j(\wp_{\Pi(m)}-d)\rangle \leq 0$, taking the limit as $m\to\infty$ in the above inequality, we have

$$\lim_{m\to\infty}\Big\|\upsilon_{\Pi(m)}-d\Big\|^2=0.$$

Thus

$$\lim_{m\to\infty}\Xi_{\Pi(m)}=\lim_{m\to\infty}\Xi_{\Pi(m)+1}=0.$$

Moreover, for all $m\geq m_0$, it is easy to notice that $\Xi_m\leq \Xi_{\Pi(m)+1}$ if $m\neq \Pi(m)$, that is, $\Pi(m) < m$, since $\Xi_i > \Xi_{i+1}$ for $\Pi(m) + 1 \leq i \leq m$. As a result, for all $m \geq m_0$, we get

$$0 \leq \Xi_m \leq max\{\Xi_{\Pi(m)}, \Xi_{\Pi(m)+1}\} = \Xi_{\Pi(m)+1}.$$

Hence, $\lim_{m\to\infty} \Xi_m = 0$. This concludes that $\{v_m\}$ converges strongly to a point d. This finishes the proof. □

Since every uniformly convex Banach space has a uniformly Gâteaux differentiable norm, our theorem can be stated in a uniformly convex Banach space, which is also uniformly smooth. Therefore, we can also obtain the following result without proof.

Corollary 3.1.2 *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{B}$ which is also uniformly smooth and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ a nonexpansive mapping such that $\mathcal{F}(\mathcal{T}) \neq \emptyset$. Let $\{v_m\}$ be a sequence generated iteratively by (3.1.1). Then $\{v_m\}$ converges strongly to a point in $\mathcal{F}(\mathcal{T})$.*

In the remainder of this section, we prove the following theorem for finding zeros of accretive mappings.

Theorem 3.1.3 *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{B}$ which has uniformly Gâteaux differentiable norm and $\Upsilon : \mathcal{C} \to \mathcal{C}$ a continuous and accretive mapping such that $N(\Upsilon) \neq \emptyset$. For arbitrary $v_0, v_1 \in \mho$, let $\{v_m\}$ be the sequence generated by*

$$\begin{cases} \hbar_m = v_m + \pi_m\ (v_m - v_{m-1}), \\ \psi_m = (1 - \xi_m)\,(1 - \sigma_m)\,\hbar_m, \\ v_{m+1} = (1 - \rho_m\ )\psi_m + \rho_m J_\Upsilon \psi_m,\ \ m \geq 1, \end{cases}$$

where $J_\Upsilon = (I+\Upsilon)^{-1}$. *Consider that the following assumptions hold:*

(i) $\lim_{m\to\infty} \xi_m = 0, \lim_{m\to\infty} \sigma_m = 0, \sum_{m=1}^{\infty} \sigma_m = \infty, \xi_m, \sigma_m \in (0,1), \rho_m \in [l_1, l_2] \subset (0,1)$,

(ii) $\pi_m \geq 0$, $\forall m \in \mathbb{N}$ *and* $\sum_{m=1}^{\infty} \pi_m < \infty$.

Then $\{v_m\}$ *converges strongly to a point in* $N(\Upsilon)$.

Proof. According to the results of Martin [20, 61, 77] and Cioranescu [27], Υ is m-accretive. This implies that $J_\Upsilon = (I+\Upsilon)^{-1}$ is nonexpansive and $\mathcal{F}(J_\Upsilon) = N(\Upsilon)$. Setting $J_\Upsilon = \mathcal{T}$ in Theorem 3.1.1. Using the same approach going forward, we obtain the desired result. □

Using the following experiment, we examine the algorithm's behavior 3.1.1 for approximating the fixed point. We show the convergence results discussed in this study graphically and with a table of numerical values.

Example 3.1.4 *Consider that a fixed point problem taken from [79] in which* $\mathcal{B} = \mathbb{R}$ *through the usual real number space* $\mathbb{R}$ *with the usual norm. A mapping* $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ *is defined by*

$$\mathcal{T}(v) = (5v^2 - 2v + 48)^{\frac{1}{3}}, \forall v \in A,$$

where $A = \{v : 0 \leq v \leq 50\}$.

Experiment 1. *For the control parameter* $\xi_m = \sigma_m = \frac{1}{(km+2)}$ *in this experiment, we used several values for* $k = 1, 2, 3, 5, 10$. *Consider* $\rho_m = 0.80, v_0 = v_1 = 10$, $\pi_m = \frac{10}{(m+1)^2}$ *and* $D_m = \|v_m - v_{m-1}\|$.

k	number of iteration (n)	elapsed time
1	449	0.013728
2	319	0.011591
3	262	0.021587
5	204	0.024854
10	145	0.036621

Table 3.1.1: The sequence generated by algorithm (3.1) while $\xi_m = \sigma_m = \frac{1}{(km+2)}$ and elapsed time for the indicated values of n.

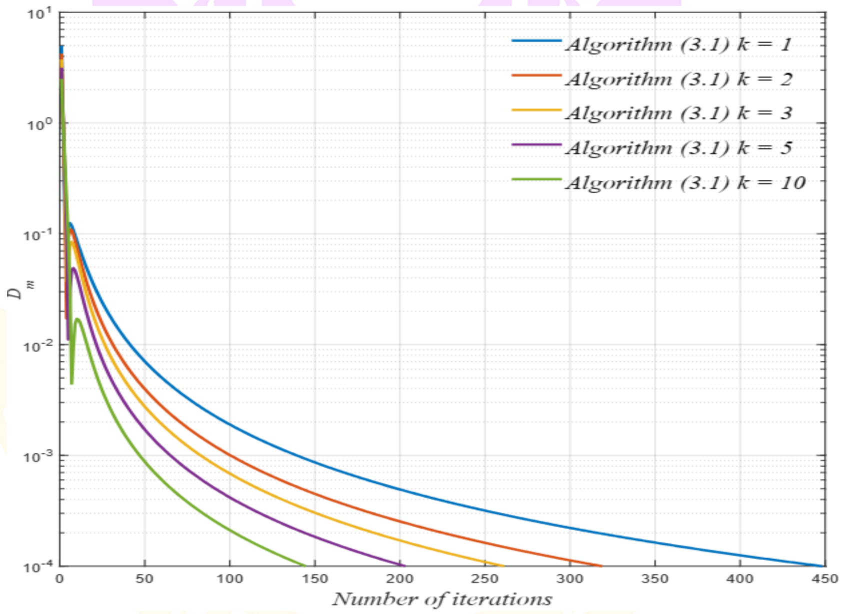


Figure 3.1.1: The convergence of algorithm (3.1) while $\xi_m = \sigma_m = \frac{1}{(km+2)}$ and the number of iteration are 449, 319, 262, 204, 145.

Experiment 2. We use several values for $k = 0.15, 0.35, 0.55, 0.75, 0.95$ *for the control parameter* $\rho_m = k$*. Also, consider* $\xi_m = \sigma_m = \frac{1}{(2m+2)}, v_0 = v_1 = 10$, $\pi_m = \frac{10}{(m+1)^2}$ *and* $D_m = \|v_m - v_{m-1}\|$.

k	number of iteration (n)	elapsed time
0.15	897	0.026490
0.35	557	0.019869
0.55	419	0.024898
0.75	336	0.028761
0.95	276	0.022688

Table 3.1.2: The sequence generated by algorithm (3.1) while $\rho_m = k$ and elapsed time for the indicated values of n.

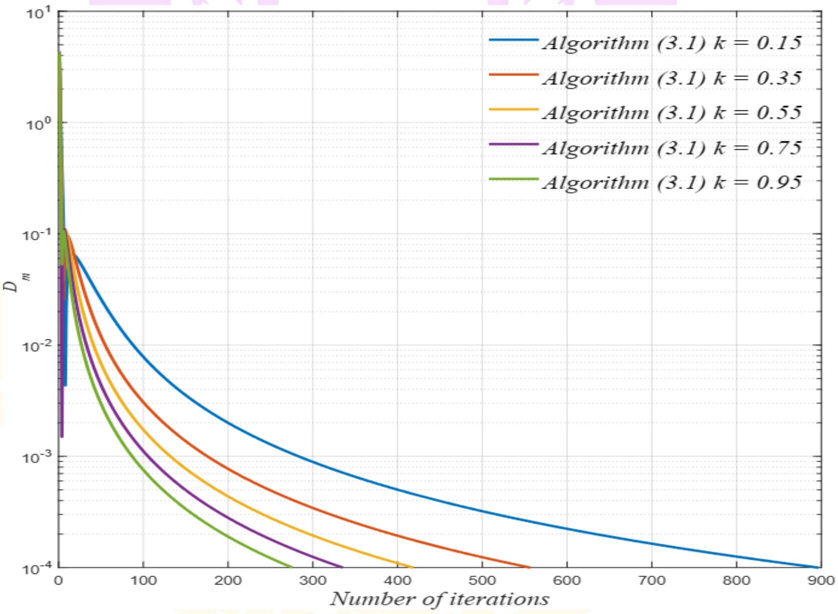


Figure 3.1.2: The convergence of algorithm (3.1) while $\rho_m = k$ and the number of iteration are 897, 557, 419, 336, 276.

Symmetry considerations can be related to signal processing, especially when signals satisfy certain symmetries. Now, we focus on applying algorithm (3.1.1) to signal recovery problems. In signal processing, compressed sensing can be modeled as the following under determined linear equation system:

$$y = Av + \nu,$$

where $v \in \mathbb{R}^n$ is the original signal with n components to be recovered, $\nu, y \in \mathbb{R}^m$ are noise and the observed signal with noise for m components, respectively, and $A \in \mathbb{R}^{m\times n}$ is a degraded matrix. Finding the solutions of the previous underdetermined linear equation system can be viewed as solving the LASSO problem:

$$\min_{v\in\mathbb{R}^N} \frac{1}{2}\|y - Av\|_2^2 + \lambda\|v\|_1,$$

where $\lambda > 0$. Various techniques and iterative schemes have been developed to solve the LASSO problem. Our method for solving the LASSO problem can be applied by setting $\mathcal{T}v = \text{prox}_{\mu g}(v - \mu\nabla f(v))$, where $f(v) = \|y - Av\|_2^2/2$, $g(v) = \lambda\|v\|_1$, and $\nabla f(v) = A^T(Av - y)$.

A straightforward observation confirms the satisfaction of all conditions in 3.1.1 Next, we conduct experiments to showcase the convergence and effectiveness of the proposed algorithm in recovering the k-sparse signal v_k recovery problem with $k = 70, 35, 18, 9$.

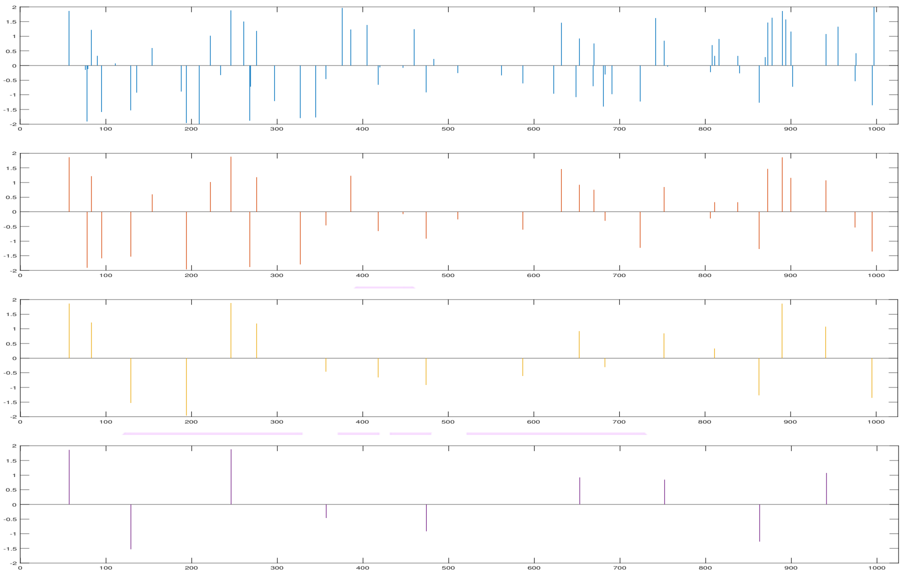


Figure 3.1.3: The k-sparse signal with $k = 70, 35, 18, 9$, respectively.

A signal of size $n = 1024$ elements, generated uniformly within the interval $[-2, 2]$, is utilized to produce observation signals $y_k = Av_k + \nu$, where $m = 512$ (see on Figure 3.1.4).

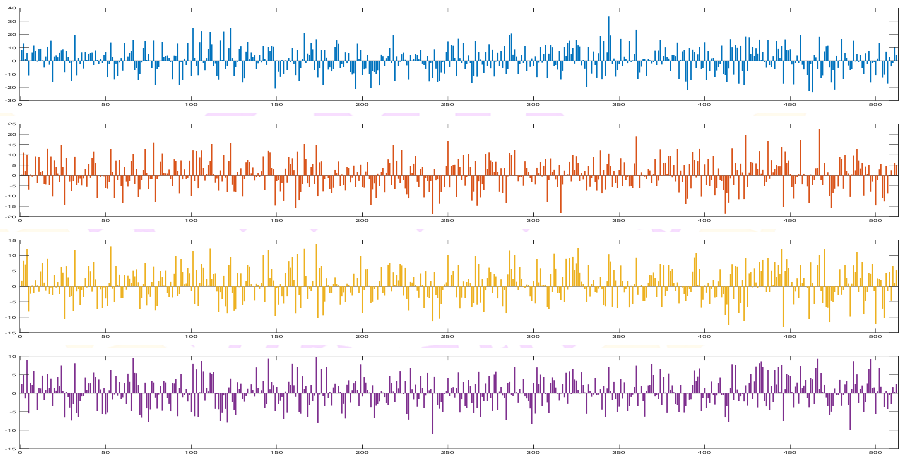


Figure 3.1.4: Degraded of k-sparse signal with $k = 70, 35, 18, 9$, respectively.

The white Gaussian noise ν is depicted in Figure 3.1.5.

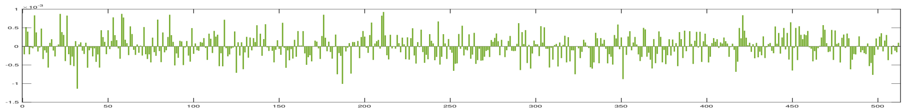


Figure 3.1.5: Noise Signal ν.

The process starts with randomly selected initial signal data v_0 and v_1, each comprising $n = 1024$ randomly chosen elements (see Figure 3.1.6).

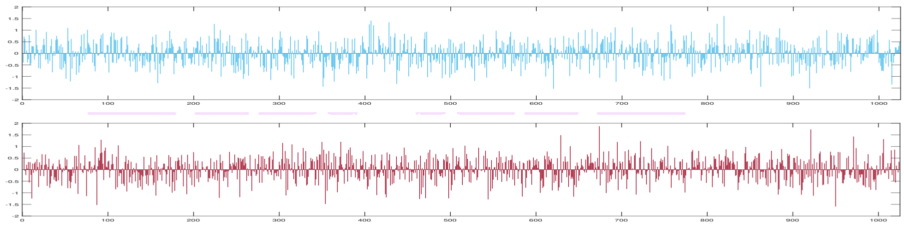


Figure 3.1.6: Initial Signals v_0 and v_1.

In addressing the challenge of recovering k-sparse signals, we reconstructed the observed signals depicted in Figure 3.1.4 to obtain the k-nonzero signal shown in Figure 3.1.3. Throughout this recovery process, we carefully considered the optimal regularization parameter, denoted as λ, to maximize the Signal-to-Noise Ratio (SNR). The performance of the proposed method at m^{th} iteration is measured quantitatively by means of the signal-to-noise ratio (SNR), which is defined by

$$\text{SNR}(v_m) = 20 \log_{10} \left(\frac{\|v_m\|_2}{\|v_m - v\|_2} \right),$$

where v_m is the recovered signal at the m^{th} iteration using the proposed method. The SNR quality influenced by the regularization parameter λ within the range $[5, 75]$, are visualized in Figure 3.1.7.

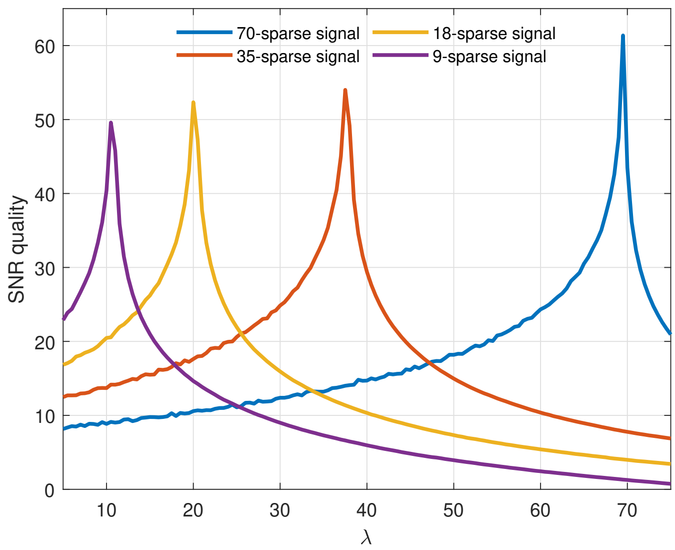


Figure 3.1.7: The plots of best SNR quality of the proposed method effected with regularize parameter λ during 1,000 iterations.

The most recent figure illustrates that the proposed algorithms can solve the sparse signal recovery challlenge. Moreover, we present the evolution of the (SNR) and relative error plot using max-norm over the number of iterations during the recovery of k-sparse signals with $k = 70, 35, 18, 9$. This is done while identifying the optimal regularization parameter, denoted as λ, to achieve the highest SNR quality, as illustrated in the figure above.

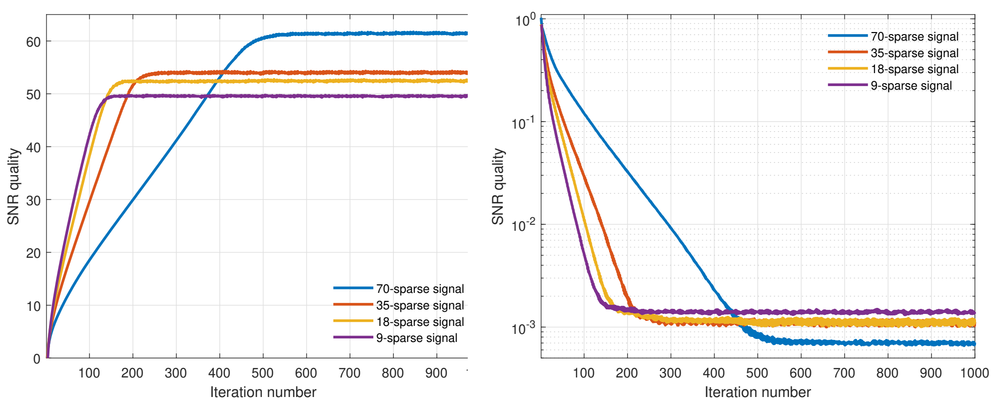


Figure 3.1.8: The SNR and relative error norm plots of the proposed algorithm effected with the optimal regularize parameter λ in recovering the observed sparse signal.

Notably, the plot of the signal's relative error exhibits a continuous decrease until it reaches convergence to a constant value. In the SNR quality plot, it is evident that the SNR value progressively rises until it stabilizes at a constant value. Additionally, the last figure demonstrate the best recovery of k-sparse signals with $k = 70, 35, 18, 9$ during 400 iterations using the proposed algorithm along with its optimal regularization parameter λ.

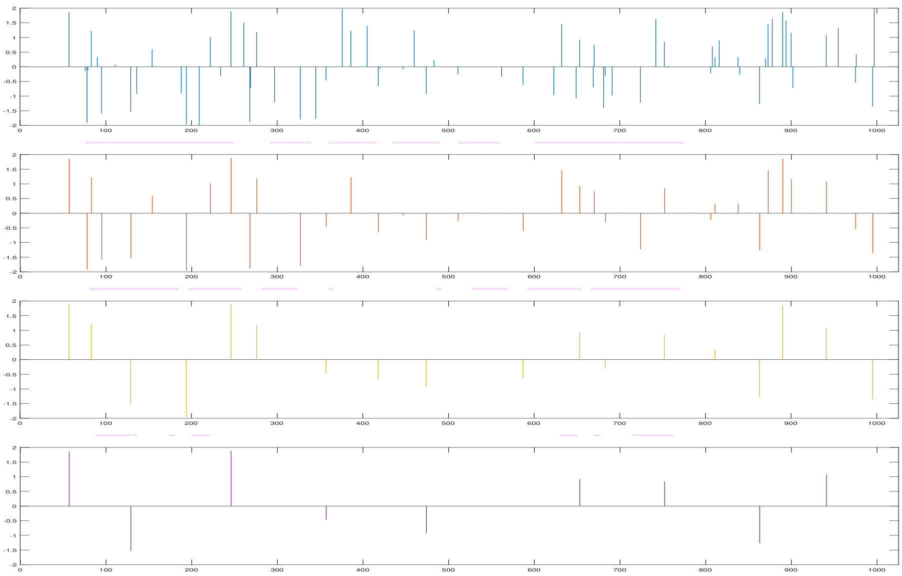


Figure 3.1.9: The best recovering of k-sparse signals with with $k = 70, 35, 18, 9$, respectively being used the proposed algorithm during 400^{th} iterations.

Based on these findings, it can be inferred that the proposed algorithm successfully enhances the quality of the recovered signal in solving the signal recovery problem.

3.2 Convergence and stability results of the Picard-SP hybrid iterative process with applications

In this section, we are now ready to prove the theorem of strong convergence of a Picard-SP hybrid iterative method (PSPHM) to a fixed point for a contraction mapping in a Banach space. We will also show that PSPHM is stable. And finally, we shall prove that PSPHM gives the faster rate of convergence than the earlier existing schemes. In addition, we also present an example using MATLAB programing. We have also given a graphical representation for this.

Theorem 3.2.1 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be the sequence generated by (1.0.12) with real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. We know that a unique $p \in F(T)$ exists (by Banach contraction theorem). We will prove that $x_n \to p$ as $n \to \infty$. Using (1.0.12) we have

$$\begin{aligned} \|w_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \|T x_n - T p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n \theta \|x_n - p\| \\ &= (1 - \gamma_n(1-\theta))\|x_n - p\|. \end{aligned} \tag{3.2.1}$$

Using (3.2.1), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)w_n + \beta_n T w_n - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n \|T w_n - T p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n \theta \|w_n - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n \|w_n - p\| \\ &= \|w_n - p\| \\ &\leq (1 - \gamma_n(1-\theta))\|x_n - p\|. \end{aligned} \tag{3.2.2}$$

From (3.2.2), we obtain

$$\|y_n - p\| = \|(1-\alpha_n)z_n + \alpha_n T z_n - p\|$$

$$\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - Tp\|$$
$$\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\theta\|z_n - p\|$$
$$\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\|$$
$$= \|z_n - p\|$$
$$\leq (1-\gamma_n(1-\theta))\|x_n - p\|. \tag{3.2.3}$$

By (3.2.3), we have

$$\|x_{n+1} - p\| = \|Ty_n - p\|$$
$$= \|Ty_n - Tp\|$$
$$\leq \theta\|y_n - p\|$$
$$\leq \|y_n - p\|$$
$$\leq (1-\gamma_n(1-\theta))\|x_n - p\|. \tag{3.2.4}$$

Let $\mu_n = \gamma_n(1-\theta)$. We observe that $\mu_n < 1$. Since $\sum_{n=1}^{\infty}\gamma_n = \infty$, using Lemma 2.1.17, we obtain $\lim_{n\to\infty}\|x_n - p\| = 0$. So, $x_n \to p$ as $n \to \infty$. This completes the proof. □

Now, we prove the stability of our iteration process (1.0.12).

Theorem 3.2.2 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Let $\{x_n\}$ be the sequence generated by (1.0.12) with real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfying $\sum_{n=1}^{\infty}\gamma_n = \infty$. Then the iterative process (1.0.12) is T-stable.*

Proof. Suppose (1.0.12) generate the sequence $x_{n+1} = f(T, x_n)$ which converges to a unique $x^* \in F(T)$ (by Theorem 3.2.1). Let $\{t_n\}$ be any sequence in C and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. We will show that $\lim_{n\to\infty}\epsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = x^*$.

Let $\lim_{n\to\infty} \epsilon_n = 0$. By using (3.2.4) we get

$$\begin{aligned}\|t_{n+1} - x^*\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - x^*\| \\ &\leq \epsilon_n + (1 - \gamma_n(1 - \theta))\|t_n - x^*\|.\end{aligned}$$

Define $a_n = \|t_n - x^*\|$, $c_n = \gamma_n(1 - \theta) \in (0, 1)$ and $b_n = \epsilon_n$, $\forall n \in \mathbb{N}$ which implies that $\frac{b_n}{c_n} \to 0$ as $n \to \infty$. Thus by the conditions of Lemma 2.1.18 we get $\lim_{n\to\infty} t_n = x^*$.

Conversely, letting $\lim_{n\to\infty} t_n = x^*$, we have

$$\begin{aligned}\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\ &\leq \|t_{n+1} - x^*\| + \|f(T, t_n) - x^*\| \\ &\leq \|t_{n+1} - x^*\| + (1 - \gamma_n(1 - \theta))\|t_n - x^*\|.\end{aligned}$$

This implies that $\lim_{n\to\infty} \epsilon_n = 0$. Hence, (1.0.12) is T-stable. The proof is completed. □

In the remainder of this section, we prove that (1.0.12) converges faster than (1.0.9) and (1.0.10) in Berindes sense.

Theorem 3.2.3 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a contraction mapping. Suppose that each of the iterative processes (1.0.9), (1.0.10) and (1.0.12) converge to the same fixed point p of T, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha \leq \alpha_n < 1$, $\beta \leq \beta_n < 1$ and $\gamma \leq \gamma_n < 1$ for some $\alpha, \beta, \gamma > 0$ and for all $n \in \mathbb{N}$. Then the Picard-SP hybrid iterative process (1.0.12) converges faster than all the other two iterative processes.*

Proof. Suppose that p is the fixed point of T. By using (1.0.12), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|Ty_n - p\| \\ &\le \theta\|y_n - p\| \end{aligned} \tag{3.2.5}$$

and

$$\begin{aligned}\|w_n-p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &\le (1-\gamma_n)\|x_n-p\| + \gamma_n\|Tx_n - Tp\| \\ &\le (1-\gamma_n)\|x_n-p\| + \gamma_n\theta\|x_n-p\| \\ &= (1-\gamma_n(1-\theta))\|x_n-p\|. \end{aligned} \tag{3.2.6}$$

In addition, using (3.2.6) and $\gamma_n(1-\theta) > 0$, we have $\|w_n-p\| \le \|x_n-p\|$. Moreover, from (3.2.6), we have

$$\begin{aligned}\|z_n-p\| &= \|(1-\beta_n)w_n + \beta_n Tw_n - p\| \\ &\le (1-\beta_n)\|w_n-p\| + \beta_n\|Tw_n - Tp\| \\ &\le (1-\beta_n)\|w_n-p\| + \beta_n\theta\|w_n-p\| \\ &\le (1-\beta_n)\|x_n-p\| + \beta_n\theta\left[(1-\gamma_n(1-\theta))\|x_n-p\|\right] \\ &= \left[1-\beta_n+\beta_n\theta(1-\gamma_n(1-\theta))\right]\|x_n-p\| \\ &= \left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|x_n-p\|. \end{aligned} \tag{3.2.7}$$

In addition, using (3.2.2) and $\gamma_n(1-\theta) > 0$, we have $\|z_n-p\| \le \|x_n-p\|$. Moreover, from (3.2.7), we have

$$\begin{aligned}\|y_n-p\| &= \|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\ &\le (1-\alpha_n)\|z_n-p\| + \alpha_n\|Tz_n - Tp\| \\ &\le (1-\alpha_n)\|z_n-p\| + \alpha_n\theta\|z_n-p\| \end{aligned}$$

$$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\theta\left[1-\beta_n(1-\theta)(1+\gamma_n\theta)\right]\|x_n - p\|$$

$$= \left[1-\alpha_n + \alpha_n\theta(1-\beta_n(1-\theta)(1+\gamma_n\theta))\right]\|x_n - p\|$$

$$= \left[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\right]\|x_n - p\|. \tag{3.2.8}$$

It follows from (3.2.5) and (3.2.8) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Ty_n - p\| \\
&\leq \theta\|y_n - p\| \\
&\leq \theta\left[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\right]\|x_n - p\| \\
&\leq \left[1-\alpha_n(1-\theta)(1+\beta_n\theta(1+\gamma_n\theta)\right]\|x_n - p\| \\
&\vdots \\
&\leq \left[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\right]^n\|x_1 - p\|.
\end{aligned}$$

Let

$$a_n = \left[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\right]^n\|x_1 - p\|.$$

Using (1.0.9), we have

$$\begin{aligned}
\|t_{n+1} - p\| &= \|(1-\alpha_n)t_n - \alpha_n Tu_n - p\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\|Tu_n - Tp\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\theta\|u_n - p\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\|u_n - p\|
\end{aligned} \tag{3.2.9}$$

and

$$\|v_n - p\| = \|(1-\gamma_n)t_n + \gamma_n Tt_n - p\|$$

$$
\begin{aligned}
&\leq (1-\gamma_n)\|t_n - p\| + \gamma_n\|Tt_n - Tp\| \\
&\leq (1-\gamma_n)\|t_n - p\| + \gamma_n\theta\|t_n - p\| \\
&= (1-\gamma_n(1-\theta))\|t_n - p\|. \qquad (3.2.10)
\end{aligned}
$$

It follows from (3.2.10) that

$$
\begin{aligned}
\|u_n - p\| &= \|(1-\beta_n)t_n + \beta_n Tv_n - p\| \\
&\leq (1-\beta_n)\|t_n - p\| + \beta_n\|Tv_n - Tp\| \\
&\leq (1-\beta_n)\|t_n - p\| + \beta_n\theta\|v_n - p\| \\
&\leq (1-\beta_n)\|t_n - p\| + \beta_n\theta\big[(1-\gamma_n(1-\theta))\|t_n - p\|\big] \\
&\leq \big[1-\beta_n + \beta_n\theta(1-\gamma_n(1-\theta))\big]\|t_n - p\| \\
&= \big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n - p\|. \qquad (3.2.11)
\end{aligned}
$$

Using (3.2.9) and (3.2.11), we have

$$
\begin{aligned}
\|t_{n+1} - p\| &\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n - p\| \\
&= \big[1-\alpha_n\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|t_n - p\| \\
&\vdots \\
&\leq \big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1 - p\|.
\end{aligned}
$$

Let

$$
b_n = \big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1 - p\|.
$$

Hence

$$
\frac{a_n}{b_n} = \frac{\big[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\big]^n\|x_1 - p\|}{\big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|t_1 - p\|} \to 0 \ \text{ as } \ n \to \infty.
$$

Therefore, the Picard-SP hybrid iterative process (1.0.12) converges faster than the Noor iterative process (1.0.9).

Now, for the sequence $\{u_n\}$ generated by (1.0.10), we have the following

$$\begin{aligned}
\|w_n - p\| &= \|(1-\gamma_n)u_n + \gamma_n T u_n - p\| \\
&\leq (1-\gamma_n)\|u_n - p\| + \gamma_n \|Tu_n - Tp\| \\
&\leq (1-\gamma_n)\|u_n - p\| + \gamma_n \theta \|u_n - p\| \\
&= (1-\gamma_n(1-\theta))\|u_n - p\|.
\end{aligned} \tag{3.2.12}$$

In addition, using (3.2.12) and $\gamma_n(1-\theta) > 0$, we have $\|w_n - p\| \leq \|u_n - p\|$. Moreover, from (3.2.12), we have

$$\begin{aligned}
\|v_n - p\| &= \|(1-\beta_n)w_n + \beta_n T w_n - p\| \\
&\leq (1-\beta_n)\|w_n - p\| + \beta_n \|Tw_n - Tp\| \\
&\leq (1-\beta_n)\|w_n - p\| + \beta_n \theta \|w_n - p\| \\
&\leq (1-\beta_n)\|u_n - p\| + \beta_n \theta \left[(1-\gamma_n(1-\theta))\|u_n - p\|\right] \\
&= \big[1 - \beta_n + \beta_n \theta (1-\gamma_n(1-\theta))\big]\|u_n - p\| \\
&= \big[1 - \beta_n(1-\theta)(1+\gamma_n\theta)\big]\|u_n - p\|.
\end{aligned} \tag{3.2.13}$$

In addition,

$$\begin{aligned}
\|v_n - p\| &\leq (1-\beta_n)\|w_n - p\| + \beta_n \theta \|w_n - p\| \\
&\leq (1-\beta_n)\|w_n - p\| + \beta_n \|w_n - p\| \\
&= \|w_n - p\| \\
&\leq \|u_n - p\|.
\end{aligned} \tag{3.2.14}$$

It follows from (3.2.13) and (3.2.14) that

$$\begin{aligned}
\|u_{n+1}-p\| &= \|(1-\alpha_n)v_n-\alpha_n Tv_n-p\| \\
&\leq (1-\alpha_n)\|v_n-p\|+\alpha_n\|Tv_n-Tp\| \\
&\leq (1-\alpha_n)\|v_n-p\|+\alpha_n\theta\|v_n-p\| \\
&\leq (1-\alpha_n)\|v_n-p\|+\alpha_n\theta\big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|u_n-p\| \\
&\leq (1-\alpha_n)\|u_n-p\|+\alpha_n\big[1-\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|u_n-p\| \\
&= \big[1-\alpha_n\beta_n(1-\theta)(1+\gamma_n\theta)\big]\|u_n-p\| \\
&\ \vdots \\
&\leq \big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|u_1-p\|.
\end{aligned}$$

Let

$$c_n=\big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|u_1-p\|.$$

Thus

$$\frac{a_n}{c_n}=\frac{\big[1-\alpha(1-\theta)(1+\beta\theta(1+\gamma\theta)\big]^n\|x_1-p\|}{\big[1-\alpha\beta(1-\theta)(1+\gamma\theta)\big]^n\|u_1-p\|}\to 0 \text{ as } n\to\infty.$$

Hence $\{x_n\}$ converges faster than $\{u_n\}$ to p. That is, the Picard-SP hybrid iterative process (1.0.12) converges faster than SP iterative process (1.0.10). □

In order to demonstrate the improved performance of the proposed PSPHM (1.0.12), we consider a numerical example in which we compare our method with the Noor (1.0.9), SP (1.0.10), and PicardNoor (1.0.11) iteration processes.

Example 3.2.4 *Let $C=[1,7]\subseteq X=\mathbb{R}$ and $T:C\to C$ be defined by $Tx=\sqrt[3]{x+6}$ for all $x\in C$. Choose $\alpha_n=\beta_n=\gamma_n=0.9$ for each $n\in\mathbb{N}$ with initial value $x_1=5$. Clearly, T is a contraction mapping and $F(T)=\{2\}$.*

The results show that each iteration scheme gets closer to the fixed point but at various speeds. Table 3.2.1 and the graphical figure 3.2.1 representation show that the PSPHM process (1.0.12) converges faster than all of the SP, Picard-Noor and Noor iterative processes, which found the fixed point in 6 iterations. The second best method is the SP iteration, which needed eight iterations, followed by the Picard-Noor iteration. The worst convergence speed is observed for the Noor iteration, which required 10 iterations to find the fixed point.

Step	PSPHM	SP	Picard-Noor	Noor
1	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
2	2.00126623535433	2.01520444639402	2.02671797466055	2.32491786957709
3	2.00000056550360	2.00008145968302	2.00024150721951	2.03524174735882
4	2.00000000025256	2.00000043657222	2.00000218331062	2.00382309861342
5	2.00000000000011	2.00000000233975	2.00000001973792	2.00041474551741
6	**2.0000000000000**	2.00000000001254	2.00000000017844	2.00004499339710
7		2.00000000000007	2.00000000000161	2.00000488108043
8		**2.0000000000000**	2.00000000000001	2.00000052952096
9			**2.0000000000000**	2.00000005744475
10				2.00000000623186

Table 3.2.1: The comparison of the convergence rates of the Noor (1.0.9), SP (1.0.10), Picard−Noor (1.0.11) and PSPHM (1.0.12) iterative processes.

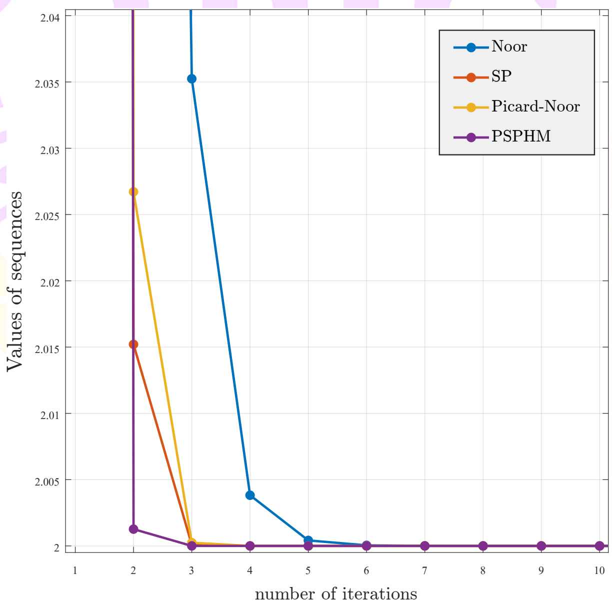


Figure 3.2.1: Convergence behavior of PSPHM (1.0.12), SP (1.0.10), Picard-Noor (1.0.11), Noor (1.0.9) iteration processes corresponding to Table 3.2.1.

Since the 1980s, the visualization patterns formed by finding polynomial roots have gained recognition, particularly in computer graphics, as noted in references [58, 82, 91]. By 2000, the term polynomiographs was coined to describe the visuals derived from these root-finding methods, with the overarching technique being polynomiography. Kalantari first introduced these concepts [50]. He defined polynomiography as both an art and a science that visualizes the approximations of zeros of complex polynomials by creating fractal and non-fractal images, capitalizing on the mathematical convergence attributes of iterative functions.

To create a polynomiograph, we first select a specific region in the complex plane, represented as $A \subset \mathbb{C}$. Within this region, each point z_0 serves as the initial point for an iterative root-finding method, noted as $R = T$ for (1.0.9), (1.0.10), (1.0.11) and (1.0.12), with n ranging from 0 to M. The iteration continues until it meets predefined convergence criteria or reaches the maximum number of iterations allowed. After the iterations conclude, we colour the initial point (z_0) using a specific colouring function. Two primary types of colouring functions are utilized:

1. Iteration-Based Coloring: This method uses a set colour map to assign colours based on the number of iterations completed.
2. Basins of Attraction Coloring: This method assigns a distinct colour to each polynomial root, and the colour assigned to a point is determined by the closest root to that point when the iterations cease.

In polynomiography, the core component of the generation algorithm is the method used for finding roots. Numerous root-finding techniques are documented in scholarly literature. Let's review some of these methods for a complex polynomial p:

1) The Newton method [51]

$$N(z) = z - \frac{p(z)}{p'(z)}.$$

2) The Halley method [51]

$$H(z) = z - \frac{2p'(z)p(z)}{2p'(z)^2 - p''(z)p(z)}.$$

3) The B_4 method (the fourth element of the Basic Family introduced by Kalantari [51])

$$B_4(z) = z - \frac{6p'(z)^2p(z) - 3p''(z)p(z)^2}{p'''(z)p(z)^2 + 6p'(z)^3 - 6p''(z)p'(z)p(z))}.$$

4) The EzzatiSaleki method (E_s for short) [37]

$$E_s(z) = N(z) + p(N(z))\left(\frac{1}{p'(z)} - \frac{4}{p'(z) + p'(N(z))}\right).$$

Visual analysis is a common technique in the contemporary examination of root-finding methods, often utilized to evaluate the stability and convergence of these methods (see [69]). This approach allows for the observation of an area rather than just a single point, providing a broader perspective on the behaviour of the method across that area, which enhances our understanding of it. In the context of polynomials, this type of visual analysis is known as polynomiography, and the individual images produced are termed polynomiographs.

In this section, we visually analyze the PSPHM, SP, Noor and Picard-Noor methods, comparing their stability and convergence against the Newton, Halley, B_4, and E_s root-finding methods.

To assess stability, we employ basins of attraction (see [3]). Within these basins, each root is assigned a unique colour, and we introduce an additional colour (black in our case) to indicate points of divergence. This colouring scheme provides insights into which root each starting point converges towards.

In our analysis, we compute specific numerical metrics, among which the convergence area index (CAI) is notably prevalent. The CAI is defined as the proportion of starting points that successfully converged to a root relative to the total number of points within the specified area, as discussed in [4]. The formula for the CAI is as follows:

$$CAI = \frac{N_c}{N},$$

where N_c is the number of points in the polynomiograph that have converged, and N is the overall count of points in the polynomiograph. The CAI values range from 0 (indicating no convergence among the points) to 1 (all points have converged). Moreover, using the polynomiograph, we can calculate an average number of iterations (ANI) (see [39]).

We use four root-finding methods in the considered example: Newton, Halley, B_4 and the E_s family. And we generate polynomiographs for a cubic polynomial $p_3(z) = z^3 - 1$, with roots: 1, $-0.5000 \pm 0.8660i$ and a complex polynomial $p_4(z) = z^4 + 4$, with roots: $\pm 1 \pm 1i$.

After each iteration, we proceed with the iteration process till the convergence test is satisfied or the maximum number of iterations is reached. The standard convergence test has the following form:

$$|x_{n+1} - x_n| < \varepsilon,$$

where $\varepsilon > 0$ is the accuracy of the computations.

The polynomiographs were generated for three different settings of values of the iterations parameters: (1) $\alpha_n = 0.01$, $\beta_n = 0.01$, $\gamma_n = 0.01$, (2) $\alpha_n = 0.5$, $\beta_n = 0.5$, $\gamma_n = 0.5$, (3) $\alpha_n = 0.95$, $\beta_n = 0.95$, $\gamma_n = 0.95$. All the other parameters needed to generate the polynomiographs were $\varepsilon = 0.001$, resolution of 100×100 pixels and K is maximum number of iterations.

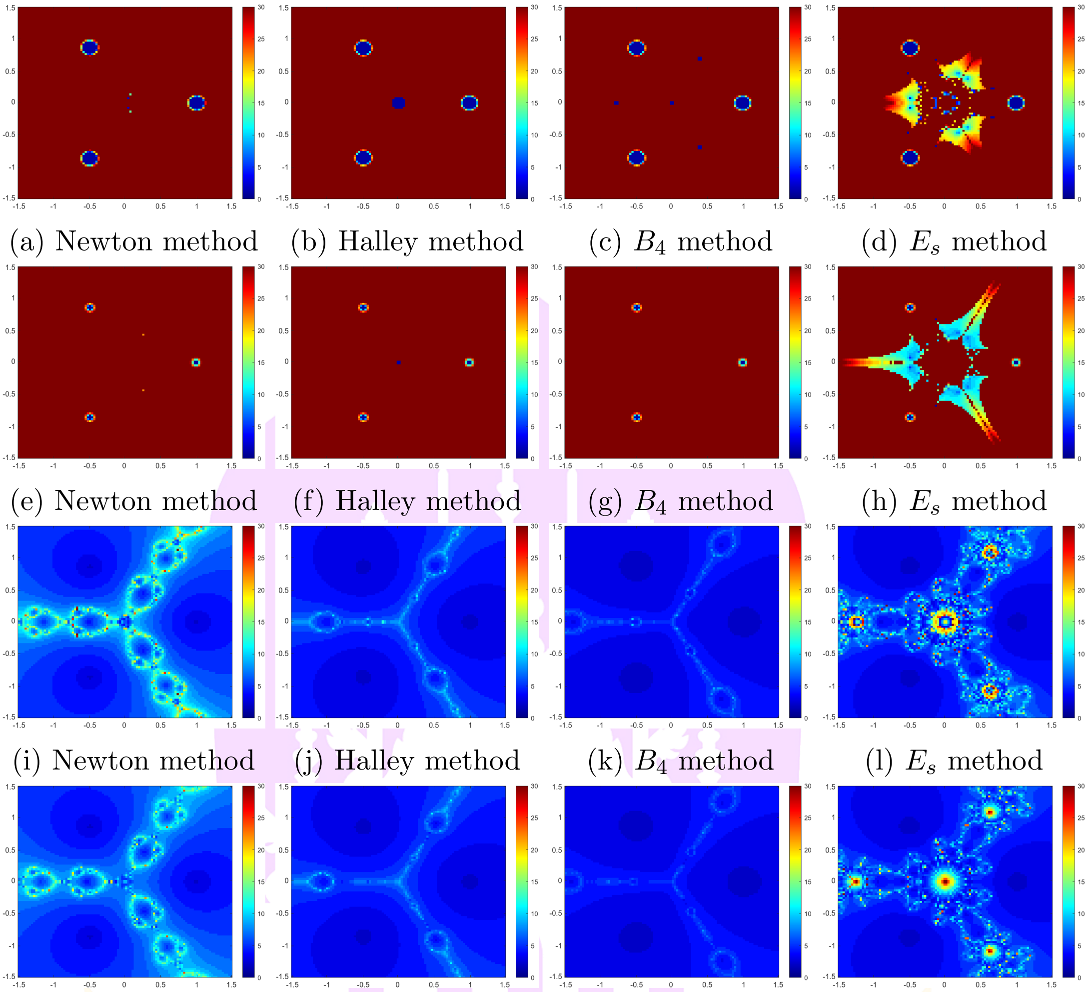
Figures 3.2.3, 3.2.7, and 3.2.11 display polynomiographs, revealing three separate basins of attraction for $p_3(z) = z^3 - 1$. Figures 3.2.5, 3.2.9, and 3.2.13 display polynomiographs, revealing four separate basins of attraction for $p_4(z) = z^4 + 4$. These basins correspond to the individual roots of the polynomial. The generated polynomiographs for the parameters in the three settings are presented in Figures 3.2.2, 3.2.6, 3.2.10, 3.2.4, 3.2.8 and 3.2.12, where as CAI and ANI values calculated from the polynomiographs are gathered in Table 3.2.2 and Table 3.2.3, respectively. (a), (b), (c), (d) come from the use of Noor iteration, (e), (f), (g), (h) come from the use of SP iteration, (i), (j), (k), (l) come from the use of Picard-Noor iteration and (m), (n), (o), (p) come from the use of PSPHM.

Iterations	Root-finding methods	$\alpha_n = \beta_n = \gamma_n = 0.01$		$\alpha_n = \beta_n = \gamma_n = 0.5$		$\alpha_n = \beta_n = \gamma_n = 0.95$	
		CAI	ANI	CAI	ANI	CAI	ANI
Noor	Newton	0.0182	29.5852	1	10.1394	1	4.2372
	Hallay	0.0214	29.5086	1	9.8398	1	3.8840
	B_4	0.0198	29.5554	1	9.8072	1	3.8356
	E_s	0.0946	28.7020	0.9620	11.682	1	4.6058
SP	Newton	0.0068	29.9084	1	5.1092	1	3.3078
	Hallay	0.0070	29.8986	1	4.6590	1	2.4776
	B_4	0.0066	29.9102	1	4.6004	1	2.2330
	E_s	0.1056	28.3330	0.9992	5.6822	1	3.1926
Picard-Noor	Newton	0.9996	6.4808	1	4.0040	1	2.8336
	Hallay	1	4.1096	1	3.0152	1	2.1314
	B_4	1	3.3986	1	2.7782	1	2.0268
	E_s	0.9982	5.2742	1	3.8536	1	2.7104
PSPHM	Newton	1	**6.0908**	1	**3.2446**	1	**2.7376**
	Hallay	1	**4.0304**	1	**2.3510**	1	**2.1270**
	B_4	1	**3.3390**	1	**2.0674**	1	**2.0264**
	E_s	0.9988	**4.8392**	1	**2.8792**	1	**2.6706**

Table 3.2.2: CAI and ANI values calculated from polynomiographs for $p_3(z) = z^3 - 1$ presented in Figures 3.2.2, 3.2.3, 3.2.6, 3.2.7, 3.2.10 and 3.2.11.

Iterations	Root-finding methods	$\alpha_n = \beta_n = \gamma_n = 0.01$		$\alpha_n = \beta_n = \gamma_n = 0.5$		$\alpha_n = \beta_n = \gamma_n = 0.95$	
		CAI	ANI	CAI	ANI	CAI	ANI
Noor	Newton	0.0460	58.4296	0.9864	10.7916	1	4.8184
	Hallay	0.0524	58.0228	1	9.7496	1	3.8660
	B_4	0.0480	58.2848	1	9.6240	1	3.7808
	E_s	0.2568	49.5092	0.8172	15.3612	0.9912	6.5176
SP	Newton	0.0548	59.1368	0.9996	5.7588	1	3.9084
	Hallay	0.0500	59.0836	1	4.6112	1	2.5200
	B_4	0.0536	59.1284	1	4.5520	1	2.2456
	E_s	0.2376	49.348	0.9516	8.6372	0.9960	5.1528
Picard-Noor	Newton	1	7.6740	1	4.4168	1	3.2688
	Hallay	1	4.1736	1	3.0624	1	2.1960
	B_4	1	3.3492	1	2.6380	1	2.0368
	E_s	0.9516	8.6372	0.9712	6.8724	0.9988	4.2688
PSPHM	Newton	1	**6.8668**	1	**3.6088**	1	**3.2008**
	Hallay	1	**4.0856**	1	**2.3796**	1	**2.1816**
	B_4	1	**3.2908**	1	**2.0960**	1	**2.0360**
	E_s	0.9912	**8.4180**	0.9936	**4.9752**	0.9988	**4.1036**

Table 3.2.3: CAI and ANI values calculated from polynomiographs for $p_4(z) = z^4 + 4$ presented in Figures 3.2.4, 3.2.5, 3.2.8, 3.2.9, 3.2.12 and 3.2.13.



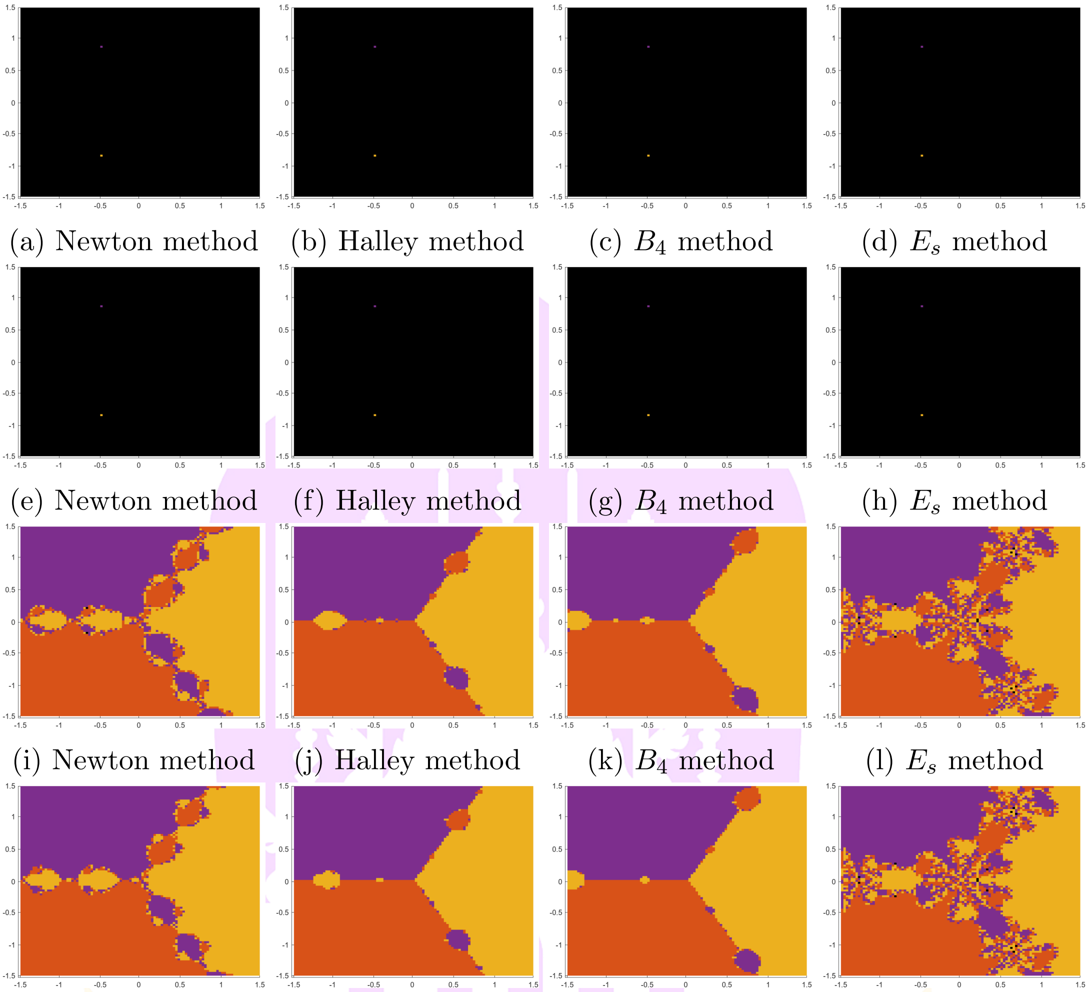
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.2: Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 30$.



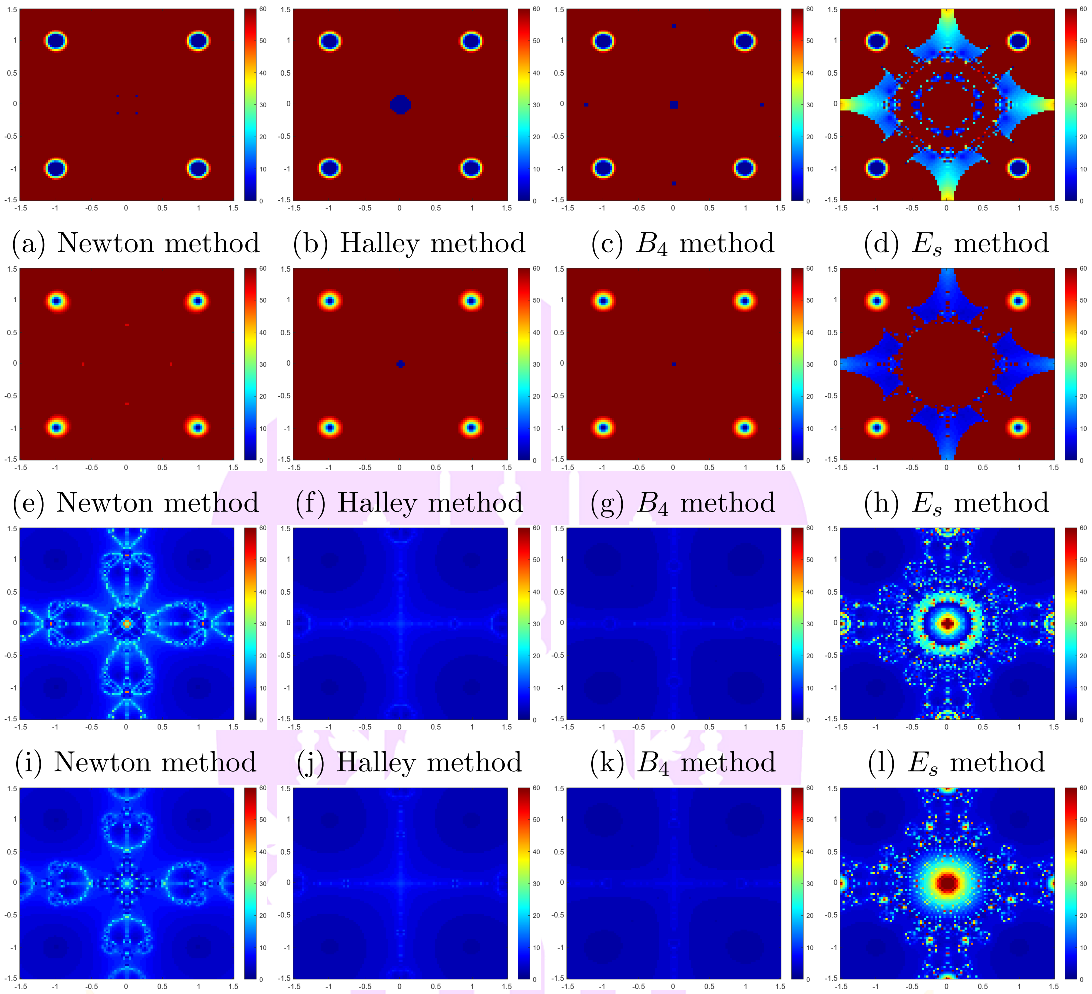
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.3: Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 30$.



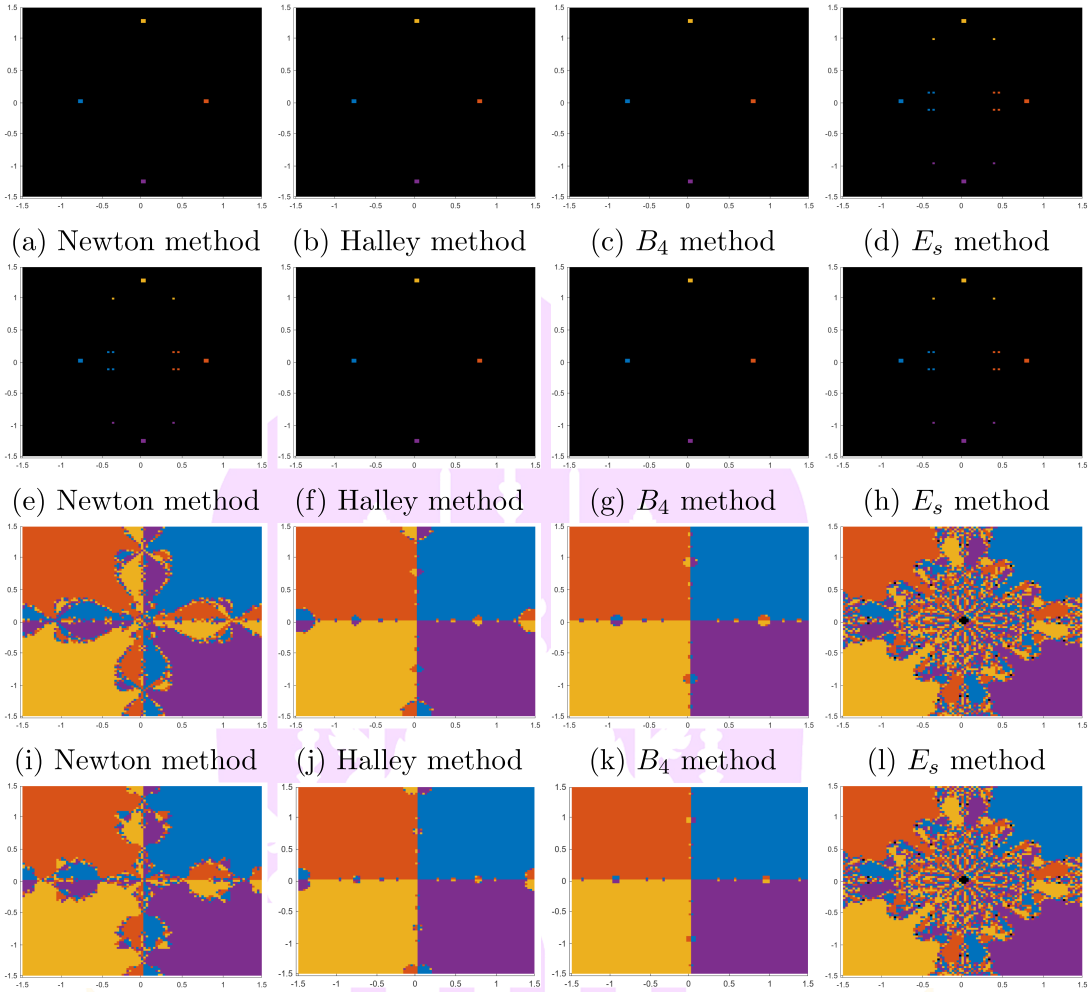
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.4: Polynomiographs for $p_4(z) = z^4 + 4$ generated using various root finding methods with the parameters $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 60$.



(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

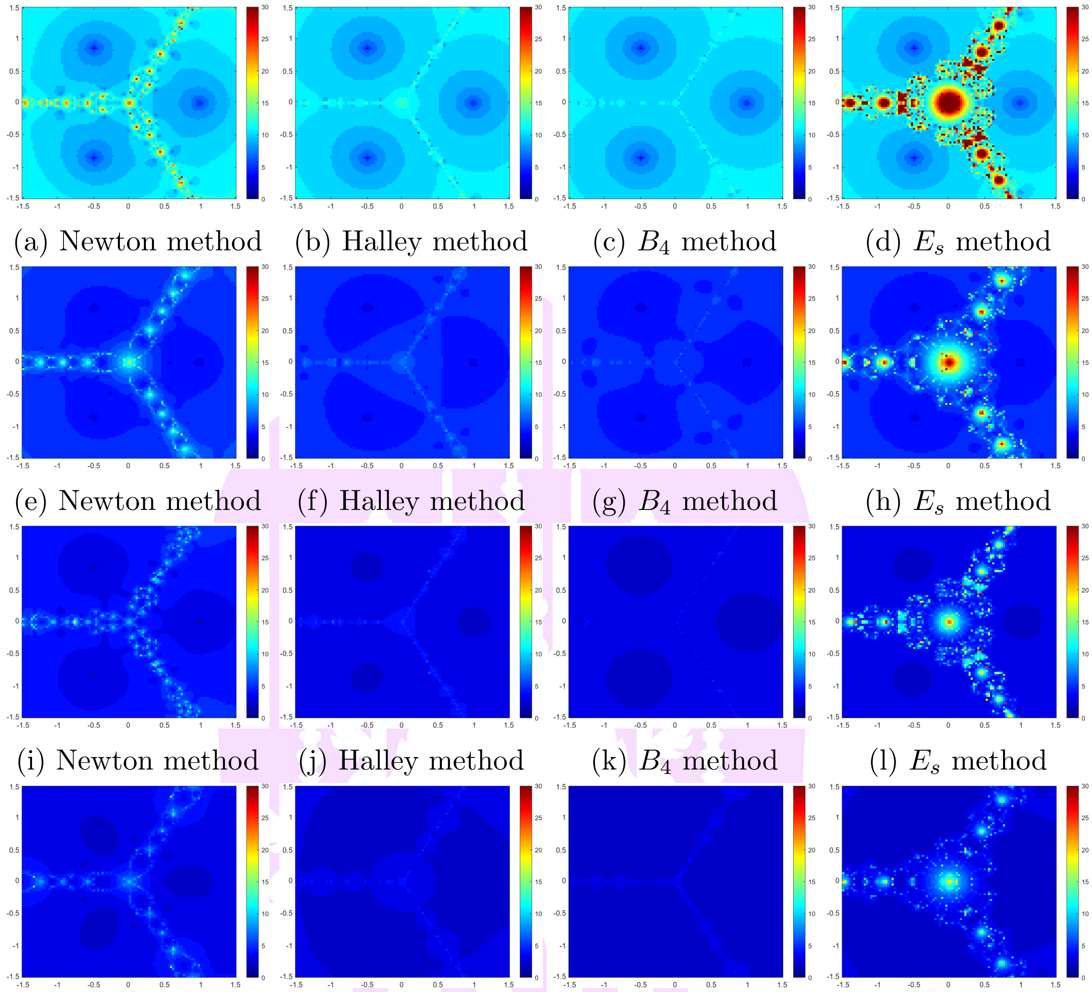
(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.5: Basins of attraction for $p_4(z) = z^4 + 4$ generated using various root finding methods with $\alpha_n = 0.01$, $\beta_n = 0.01$ and $\gamma_n = 0.01$ for $K = 60$.

For low values of the parameters ($\alpha_n = 0.01$, $\beta_n = 0.01$, $\gamma_n = 0.01$), we see that two of the iterations (Noor and SP) have not converged to any of the roots of $p_3(z)$ (see Figure 3.2.2) and $p_4(z)$ (see Figure 3.2.4), i.e., we see a uniform red colour, which corresponds to the maximal of iterations. We see a different speed convergence for the other two iterations (Picard-Noor and PSPHM). Based on the visual analysis, we can observe that the fastest convergence speed is obtained by the proposed PSPHM, followed by the Picard-Noor iteration. The ANI values confirm these observations in Table 3.2.2 and Table 3.2.3. In Table 3.2.2, the lowest ANI value, 3.3390 for the B_4, is obtained by the PSPHM, followed by the Picard-Noor (3.3986) iteration. Furthermore, it is noteworthy that the ANI values for PSPHM, which are 6.0908, 4.0340, 3.3390, and 4.8392 for the Newton, Halley, B_4, and E_s methods, respectively, also yield better results than the ANI values of the Picard-Noor, SP, and Noor iterations. Similarly, in Table 3.2.3, the lowest ANI value, 3.2908 for the B_4, is obtained by the PSPHM, followed by the Picard-Noor (3.3492) iteration. Tables 3.2.2 and 3.2.3 also found that the ANI values for the Noor and SP iterations for the Newton, Halley, B_4, and E_s methods are relatively high compared to the Picard-Noor and PSPHM iterations.

Upon analyzing the polynomiographs presented in Figure 3.2.3 and Figure 3.2.5, we see that the best stability in finding the roots is the PSPHM. This phenomenon is especially pronounced in the PSPHM mode for the Newton, Halley and B_4 methods, where CAI achieves a perfect score of 1 (see Tables 3.2.2 and 3.2.3). While characteristic braids are visible in each case, their shapes vary among the methods. The most intricate braids are observed with the E_s method, resulting in the largest interweaving of basins. Except in Noor and SP iterations, a small percentage of the starting points did not converge to any roots (black colour indicate points of divergence).



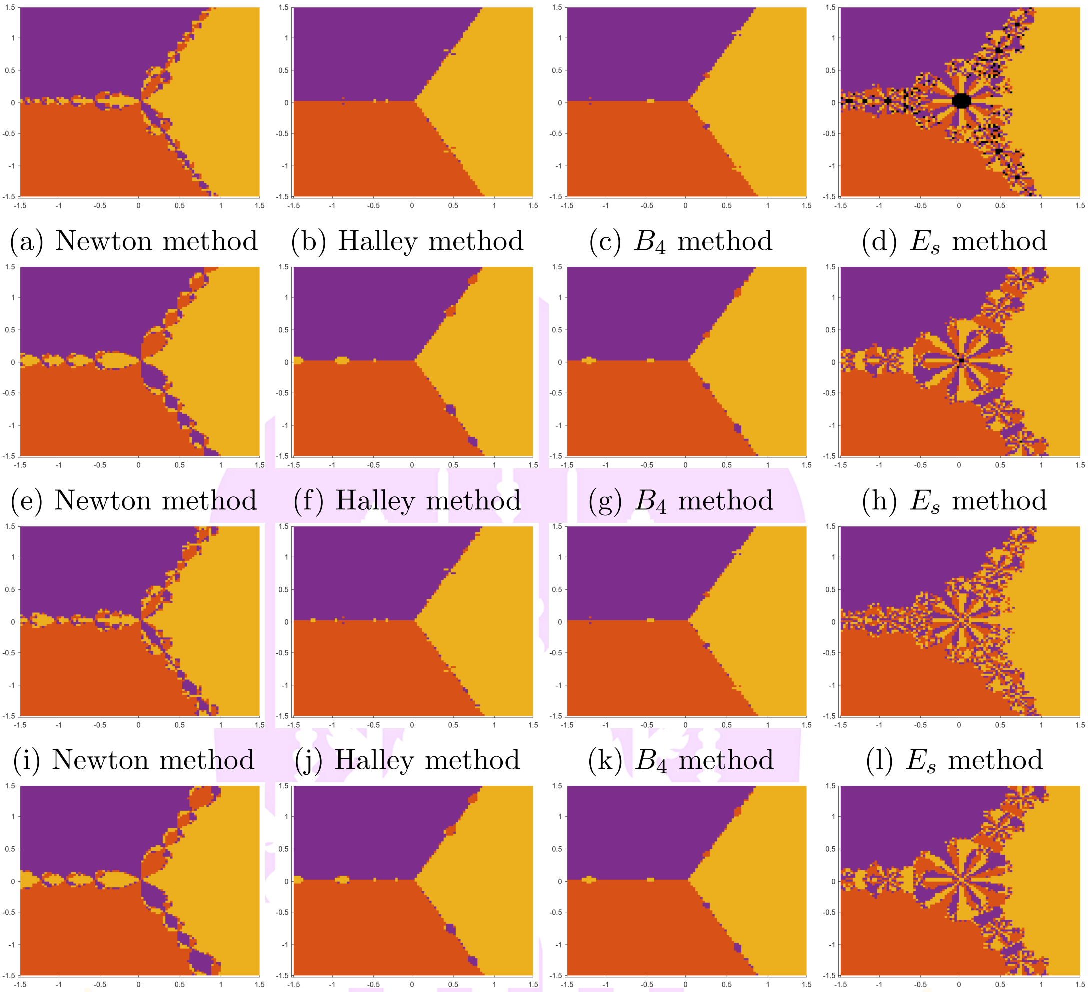
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.6: Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 30$.



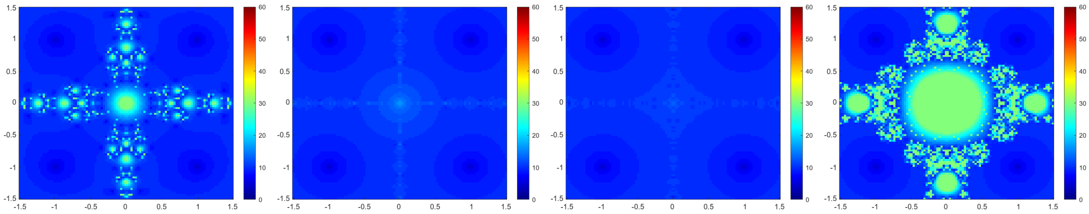
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

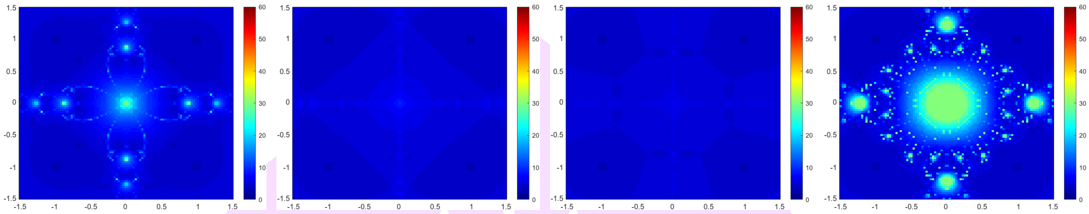
(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

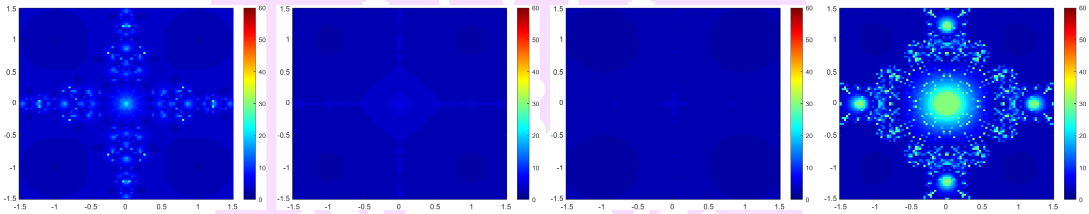
Figure 3.2.7: Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 30$.



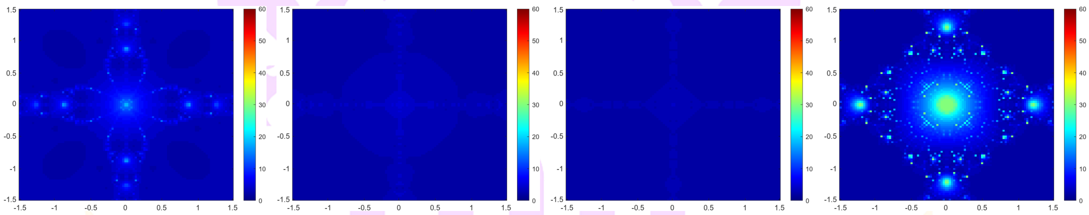
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method



(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

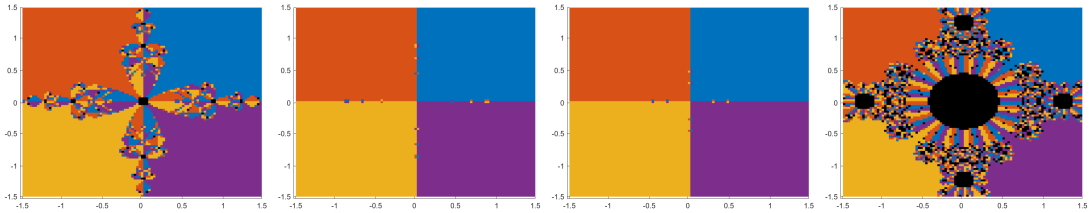


(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

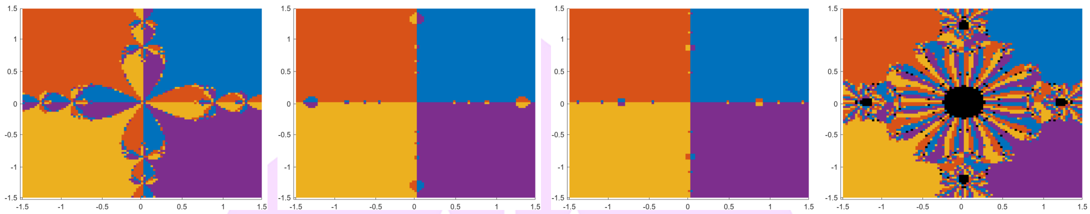


(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

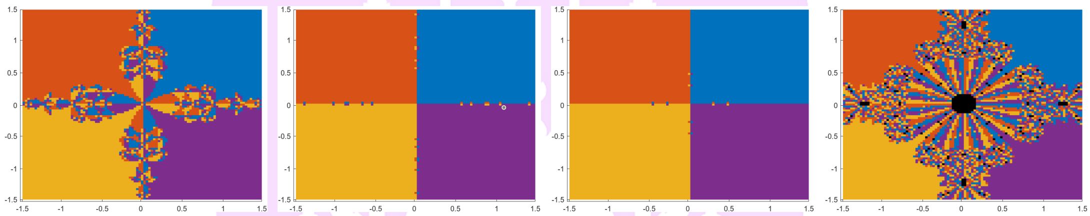
Figure 3.2.8: Polynomiographs for $p_4(z) = z^4 + 4$ generated using various root finding methods with the parameters $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 60$.



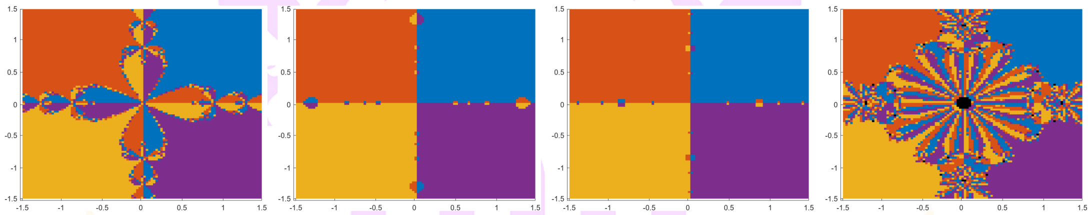
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method



(e) Newton method (f) Halley method (g) B_4 method (h) E_s method



(i) Newton method (j) Halley method (k) B_4 method (l) E_s method



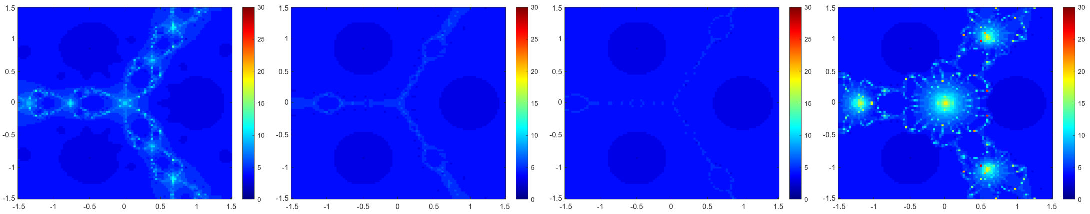
(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.9: Basins of attraction for $p_4(z) = z^4 + 4$ generated using various root finding methods with $\alpha_n = 0.5$, $\beta_n = 0.5$ and $\gamma_n = 0.5$ for $K = 60$.

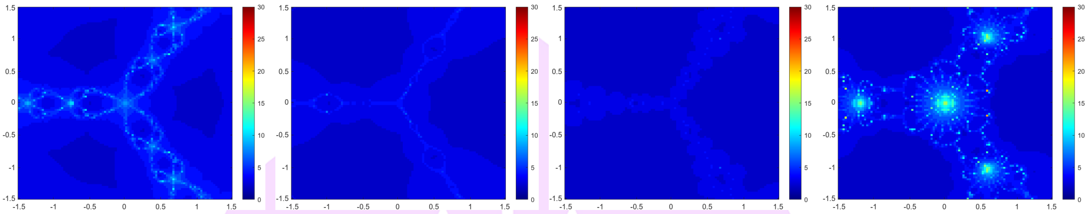
For polynomiographs for the second parameters setting ($\alpha_n = 0.5$, $\beta_n = 0.5$, $\gamma_n = 0.5$) presented in Figure 3.2.6 and Figure 3.2.8, we see that the Noor iteration obtains the slowest speed of convergence. In Figure 3.2.6, the polynomiograph contains red colours, indicating a high number of performed iterations. When we look at the polynomiographs presented in Figure 3.2.6 and Figure 3.2.8, we see that the fastest among the analyzed iterations is the PSPHM. In the polynomiographs, we can observe darker blue colours than in the case of the other iteration processes, which shows a smaller number of performed iterations.

The ANI values in Table 3.2.2 and Table 3.2.3 confirm this observation because the PSPHM obtains the lowest value. The second best iteration, in terms of convergence speed, is the Picard-Noor iteration, followed by SP and Noor iterations. The lowest value of ANI equal to 2.0674 (Table 3.2.2) and 2.0960 (Table 3.2.3) for the B_4 are obtained by the PSPHM.

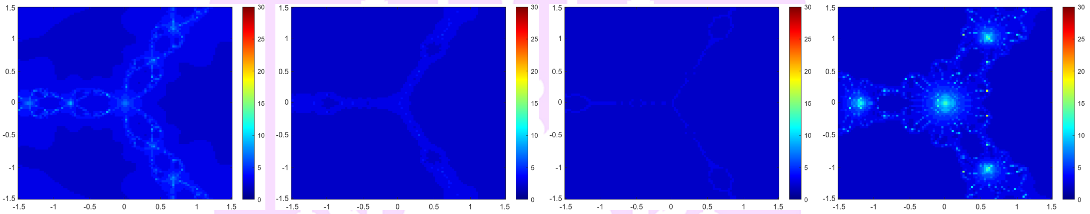
In Figure 3.2.8 and Figure 3.2.11, the interweaving of the basins around the braids is minimal for the Halley and B_4 methods, and the braids appear similar in Noor, SP, Picard-Noor and PSPHM iterations. Outside the braided regions, the behaviour of the methods is quite similar. Thus, the Halley and B_4 methods exhibit the most stable behaviour. We can also observe this by looking at the values of CAI in Tables 3.2.2 and 3.2.3. Regarding CAI value in Table 3.2.2, the best two methods were Picard-Noor and PSPHM, which obtained convergence of all starting points, i.e., CAI value equal to 1.



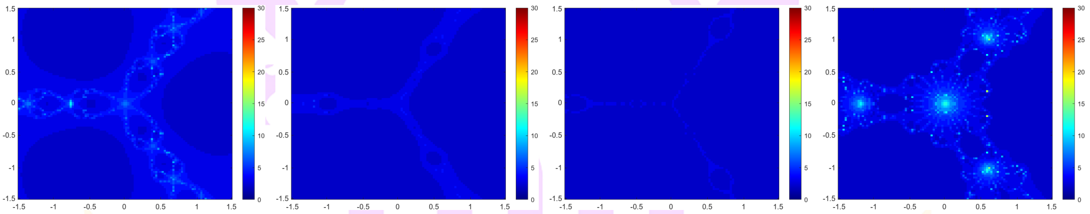
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method



(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

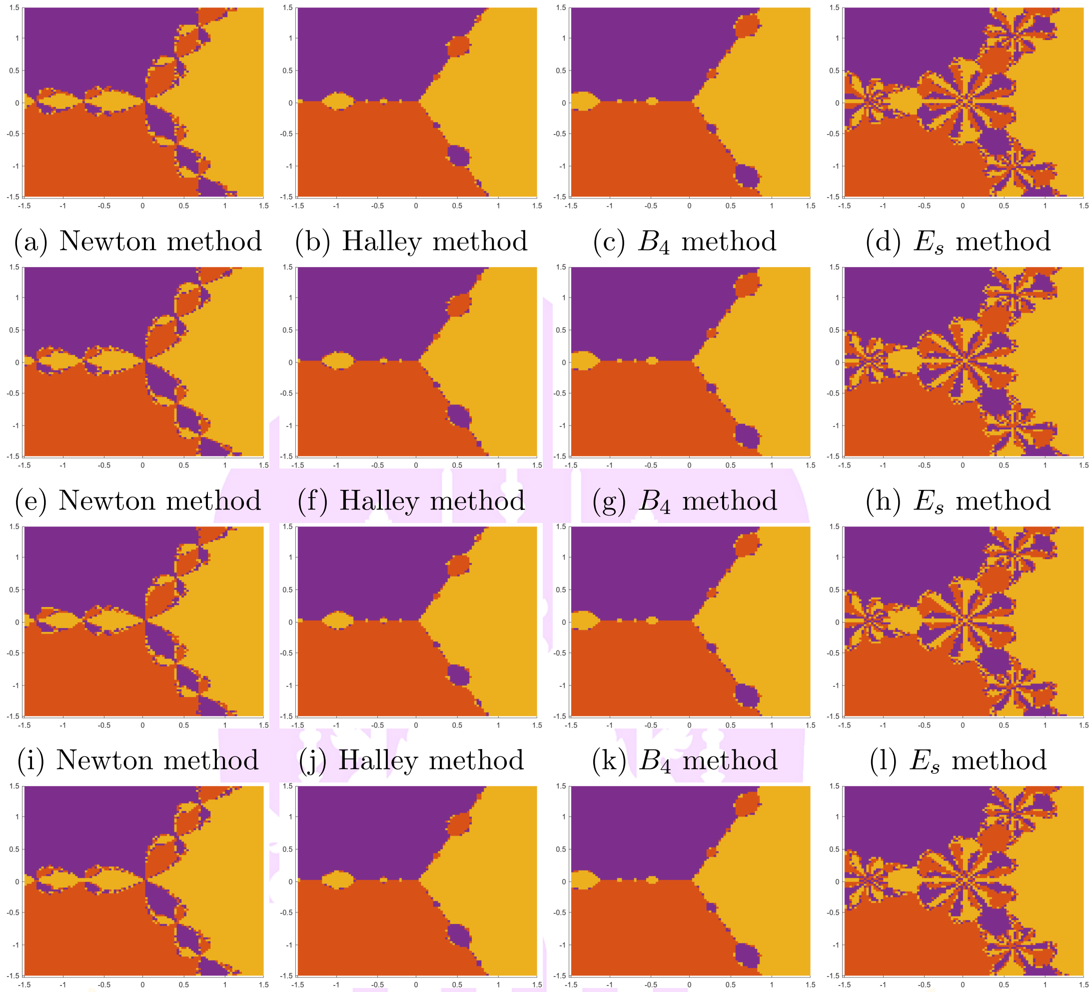


(i) Newton method (j) Halley method (k) B_4 method (l) E_s method



(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.10: Polynomiographs for $p_3(z) = z^3 - 1$ generated using various root finding methods with the parameters $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 30$.



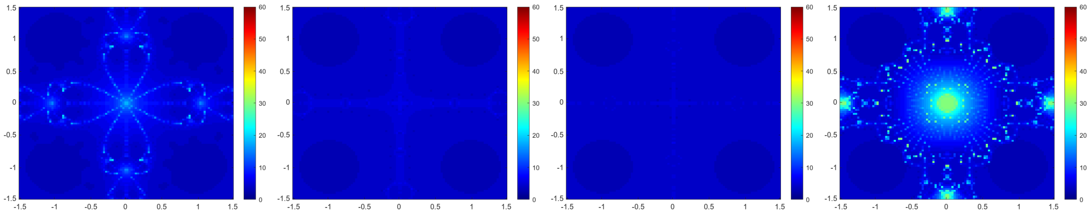
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method

(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

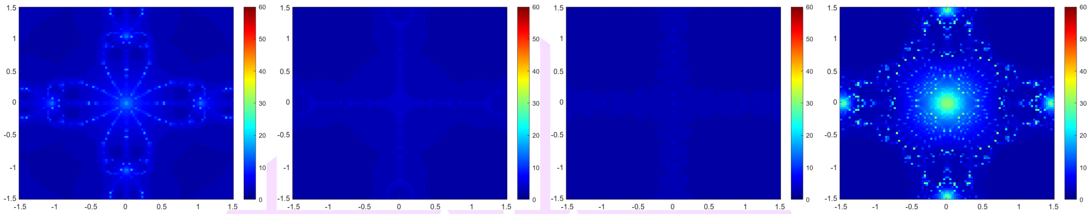
(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

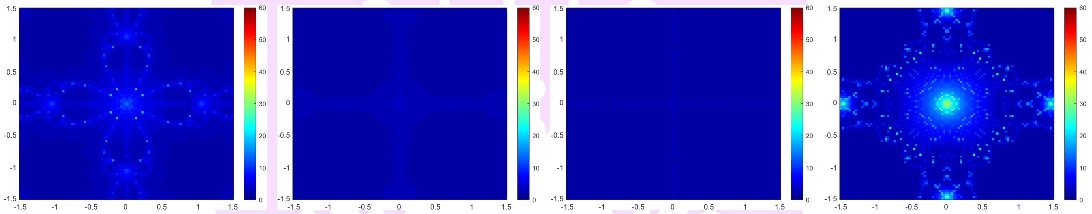
Figure 3.2.11: Basins of attraction for $p_3(z) = z^3 - 1$ generated using various root finding methods with $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 30$.



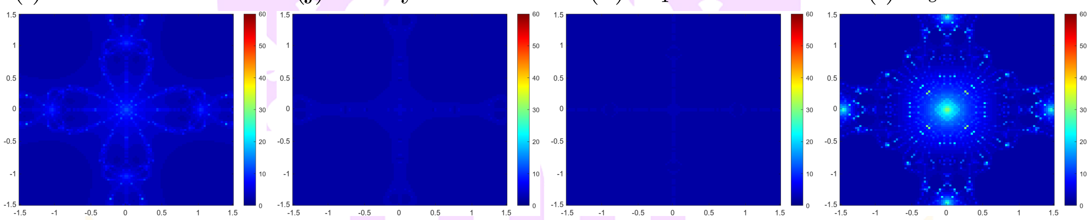
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method



(e) Newton method (f) Halley method (g) B_4 method (h) E_s method

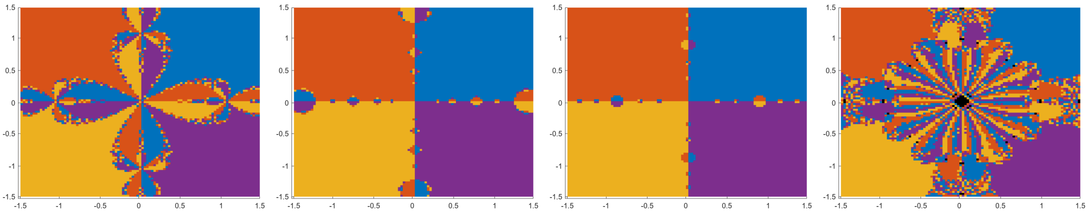


(i) Newton method (j) Halley method (k) B_4 method (l) E_s method

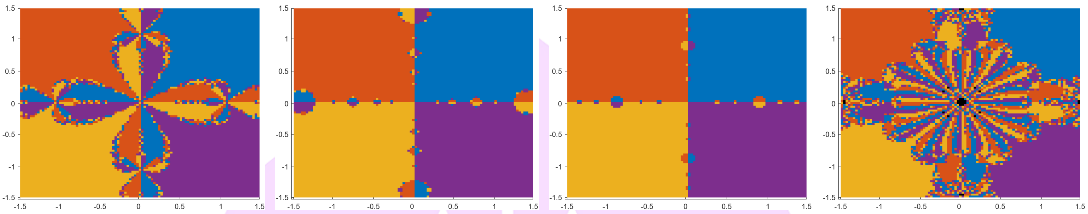


(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

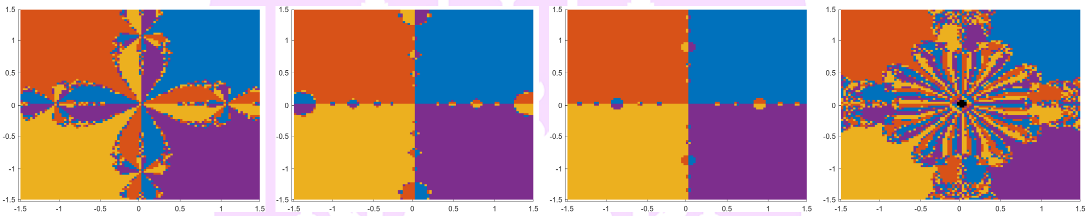
Figure 3.2.12: Polynomiographs for $p_4(z) = z^4 + 4$ generated using various root finding methods with the parameters $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 60$.



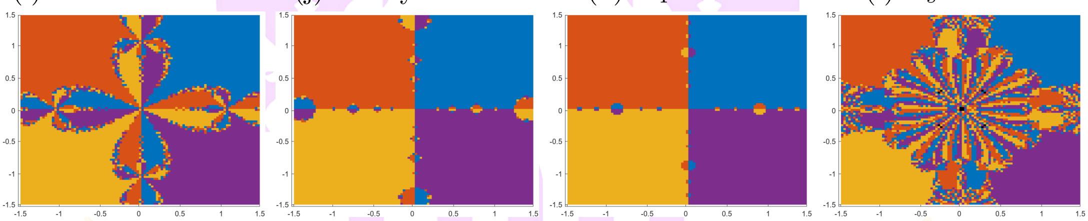
(a) Newton method (b) Halley method (c) B_4 method (d) E_s method



(e) Newton method (f) Halley method (g) B_4 method (h) E_s method



(i) Newton method (j) Halley method (k) B_4 method (l) E_s method



(m) Newton method (n) Halley method (o) B_4 method (p) E_s method

Figure 3.2.13: Basins of attraction for $p_4(z) = z^4 + 4$ generated using various root finding methods with $\alpha_n = 0.95$, $\beta_n = 0.95$ and $\gamma_n = 0.95$ for $K = 60$.

In the last parameter setting, we use high values of the parameters ($\alpha_n = 0.95$, $\beta_n = 0.95$, $\gamma_n = 0.95$). Like for the other two parameter settings, for the polynomiographs, we see that the Noor iteration obtains the slowest speed of convergence. On the other hand, the PSPHM again obtains the fastest convergence speed. In the case of each polynomiograph, we can observe darker blue colours than for the two other parameter settings, which shows a smaller number of performed iterations. This shows that for higher values of the parameters, all the iterations need fewer iterations to find the roots. We can also observe this by looking at the values of ANI in Tables 3.2.2 and 3.2.3. We see that the PSPHM obtains the lowest ANI value for high values of the parameters. The lowest values for the other iterations are also obtained for high values of the parameters.

Examining the values in Table 3.2.2, we find that CAI indicates the best performance for every method in Noor, SP, Picard-Noor and PSPHM iterations. It achieves convergence for all starting points within the area, with a CAI value 1. In Table 3.2.3, the E_s method demonstrated a favourable convergence ratio with CAI values ranging from 0.9912, 0.996 to 0.9988 for Noor, SP, Picard-Noor and PSPHM iterations. This also indicates that there were instances where a small portion of the initial points did not converge to any of the roots. It is noted that although the Halley, B_4, and Newton methods give good CAI values, the E_s method produces beautiful images with more artistic value.

CHAPTER IV

CONCLUSIONS

4.1 Conclusion

The following results are all main theorems of this thesis:

Theorem 4.1.1 *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{B}$ which has uniformly Gâteaux differentiable norm and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ a nonexpansive mapping such that $\mathcal{F}(\mathcal{T}) \neq \emptyset$. Consider that the following assumptions hold:*

(i) $\lim_{m\to\infty} \xi_m = 0$, $\lim_{m\to\infty} \sigma_m = 0$, $\sum_{m=1}^{\infty} \sigma_m = \infty$, $\xi_m, \sigma_m \in (0,1), \rho_m \in [l_1, l_2] \subset (0,1)$,

(ii) $\pi_m \geq 0$, $\forall m \in \mathbb{N}$ *and* $\sum_{m=1}^{\infty} \pi_m < \infty$.

For arbitrary $\nu_0, \nu_1 \in \mathcal{C}$. Let $\{v_m\}$ be the sequence generated by

$$\begin{cases} \hbar_m = v_m + \pi_m \,(v_m - v_{m-1}), \\ \psi_m = (1-\xi_m)\,(1-\sigma_m)\,\hbar_m, \\ v_{m+1} = (1-\rho_m\,)\psi_m + \rho_m \mathcal{T}\psi_m, \;\; m \geq 1. \end{cases} \tag{4.1.1}$$

Then $\{v_m\}$ converges strongly to a point in $\mathcal{F}(\mathcal{T})$.

Theorem 4.1.2 *Let $\mathcal{C}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{B}$ which has uniformly Gâteaux differentiable norm and $\Upsilon : \mathcal{C} \to \mathcal{C}$ a continuous and accretive mapping such that $N(\Upsilon) \neq \emptyset$. For arbitrary*

$v_0, v_1 \in \mho$, *let* $\{v_m\}$ *be the sequence generated by*

$$\begin{cases} \hbar_m = v_m + \pi_m \, (v_m - v_{m-1}), \\ \psi_m = (1 - \xi_m)\,(1 - \sigma_m)\,\hbar_m, \\ v_{m+1} = (1 - \rho_m \,)\psi_m + \rho_m J_\Upsilon \psi_m, \;\; m \geq 1, \end{cases}$$

where $J_\Upsilon = (I + \Upsilon)^{-1}$. *Consider that the following assumptions hold:*

(i) $\lim_{m\to\infty} \xi_m = 0, \lim_{m\to\infty} \sigma_m = 0, \sum_{m=1}^{\infty} \sigma_m = \infty, \xi_m, \sigma_m \in (0,1), \rho_m \in [l_1, l_2] \subset (0,1)$,

(ii) $\pi_m \geq 0$, $\forall m \in \mathbb{N}$ *and* $\sum_{m=1}^{\infty} \pi_m < \infty$.

Then $\{v_m\}$ *converges strongly to a point in* $N(\Upsilon)$.

Theorem 4.1.3 *Let* C *be a nonempty closed convex subset of a Banach space* X *and* $T : C \to C$ *be a contraction mapping. Let* $\{x_n\}$ *be the sequence generated by (1.0.12) with real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfying* $\sum_{n=1}^{\infty} \gamma_n = \infty$. *Then* $\{x_n\}$ *converges strongly to a unique fixed point of* T.

Theorem 4.1.4 *Let* C *be a nonempty closed convex subset of a Banach space* X *and* $T : C \to C$ *be a contraction mapping. Let* $\{x_n\}$ *be the sequence generated by (1.0.12) with real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfying* $\sum_{n=1}^{\infty} \gamma_n = \infty$. *Then the iterative process (1.0.12) is* T*-stable.*

Theorem 4.1.5 *Let* C *be a nonempty closed convex subset of a Banach space* X *and* $T : C \to C$ *be a contraction mapping. Suppose that each of the iterative processes (1.0.9), (1.0.10) and (1.0.12) converge to the same fixed point* p *of* T, *where* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha \leq \alpha_n < 1$, $\beta \leq \beta_n < 1$ *and* $\gamma \leq \gamma_n < 1$ *for some* $\alpha, \beta, \gamma > 0$ *and for all* $n \in \mathbb{N}$. *Then the Picard-SP hybrid iterative process (1.0.12) converges faster than all the other two iterative processes.*

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Publications

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Conference presentations

1. **Baewnoi, K.** (August 2 - 5, 2023). Modified three-step algorithms with an inertial technique for nonexpansive mappings with an application. The 11 th Asian Conference on Fixed Point Theory and Optimization 2023 (ACFPTO2023), Pattaya, Thailand.

2. **Baewnoi, K.** (May 29 - 31, 2024). Convergence analysis and polynomiographic visualization of Picard-SP hybrid iterative methods. The 28th Annual Meeting in Mathematics (AMM2024),

Ubon Ratchathani University.

3. **Baewnoi, K.** (Augustr 6 - 8, 2024). Convergence and stability results of the Picard-P hybrid iterative process with applications. The 4th International Conference and Workshop on Applied Nonlinear Analysis (ICWANA 2024), Bangsaen Heritage Hotel, Bangsean, Chonburi, Thailand.

4. **Baewnoi, K.** (January 15 - 18, 2025). Convergence and visual analysis of the Picard-D hybrid iterative process. The 12th Asian Conference on Fixed Point Theory and Optimization 2025 (ACFPTO2025), Duangtawan Hotel Chiang Mai, Chiang Mai, Thailand